Cornell University Mathematics Department Senior Thesis

The Weak Order of Coxeter Systems and Combinatorial Properties of Descent Sets

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Abstract

In this paper, we study the weak order of Coxeter systems and the combinatorial properties of descent sets. There are three main results: (1) Given a Coxeter system (W, S), some word $v \in W$, some subset $A \subseteq S$ disjoint from $D_R(v)$, and some $w \in W_A$, we proved that $D_R(vw) \subseteq D_R(v) \cup A$. (2) We obtained an explicit map for $A \cup B$ to dominate B in the case when A, B are commuting disjoint sets, with B finite. (3) We proved that for finite Coxeter systems (W, S), with subsets $A, B \subseteq S$, if A dominates B, then $B \subseteq A$. In particular, the third result is a generalization of a proposition in [K. Nyman, E. Swartz, Inequalities for the h-vectors and flag h-vectors of geometric lattices, Discrete Comput. Geom. 32 (2004) 533-548], while the second result gives a partial answer to one of the problems posed in [E. Swartz, g-elements, finite buildings and higher Cohen-Macaulay connectivity, J. Combin. Theory Ser. A 113 (2006) 1305-1320]. Also, this paper develops the theory of sequences of braid moves, boundary pairs, and tagging letters in reduced expressions for the general Coxeter system (W, S). A side application of inversion tables also yield an explicit formula of a reduced expression for all words in Coxeter systems of type A_n. All these results are new.

Acknowledgements

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Also, I would like to thank my parents for always being there for me. Thank you for believing in me. I thank my friends too for being understanding and giving me the support I need.

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Introduction

As the title suggests, this paper is about the weak order of Coxeter systems and the combinatorial properties of descent sets. However, the motivation behind this paper is very different in nature, and in this introduction, we shall first give a brief historical overview. All terms and notations in this introduction will be defined in later chapters.

In 1997, Chari introduced the notion of a convex ear decomposition [Cha97] and proved that if Δ is a $(d-1)$-dimensional simplicial complex with a convex ear decomposition, then the h-vector of Δ must satisfy the inequalities $h_{i-1} \leq h_i$ and $h_i \leq h_{d-i}$ for all $i \leq \lfloor \frac{d}{2} \rfloor$. In 2004, Nyman and Swartz [NS04] proved that the order complex of a geometric lattice has a convex ear decomposition, hence Chari's result implies the h-vector inequalities for geometric lattices. Using Björner's result, which states that the flag h-vector $h_S(P)$ of a graded poset P admitting an R-labelling counts the number of maximal chains of P with labels having descent set S (See Theorem 2.7, [Bjo80]), Nyman and Swartz also proved (in the same paper [NS04]) that given sets A and B, A dominates B implies the flag h-vector inequality $h_B \leq h_A$ for all geometric lattices.

In 2005, the work in [NS04] was slightly extended by DeVries, a former Cornell student, as part of his senior thesis project [DeV05]. DeVries worked on finding explicit injections for A to dominate B, and trying two different approaches, he showed that both approaches do not yield the desired injections. DeVries also proved two special cases of Conjecture 5.7 in [NS04], and verified the original conjecture for all cases $r \leq 9$, the previous record being $r \leq 8$ in [NS04].

In 2006, Schweig studied the convex ear decompositions of posets in relation to the flag h-vectors as part of his PhD thesis in [Sch08]. He proved that the order complex of a rank-selected subposet of a geometric lattice admits a convex ear decomposition, hence extending Nyman and Swartz's result in [NS04]. Schweig also proved that the rank-selected subposets of supersolvable lattices with nowhere-zero Möbius function and the rank-selected subposets of face posets of Cohen-Macaulay simplicial complexes have order complexes that admit convex ear decompositions [Sch08]. Consequently, applying Chari's result [Cha97], he obtained the flag h-vector inequalities analogous to those obtained in [NS04]. These inequalities involve descent sets and the notion of A dominating B for sets A and B.

Also in 2006, Swartz [Swa06] studied finite buildings and proved that if Δ is a finite building of type (W, S), and if $A, B \subseteq S$ such that A dominates B, then $h_B \leq h_A$. Again, we get a connection between descent sets of Coxeter systems and another area, this time being the theory of finite buildings.

Chari's result on convex ear decomposition relies on a deep result by Stanley [Sta80], which involves the hard Lefschetz Theorem from algebraic geometry. This means the above results involving inequalities of the h-vector are all indirectly dependent on the Lefschetz Theorem. It would then be very desirable to be able to give a combinatorial proof to the inequalities of the flag h-vector and avoid using the Lefschetz Theorem, hence providing a combinatorial proof to all the above-mentioned results.

Motivated by the results on dominating sets in [NS04], we study the descent sets of general Coxeter systems, hoping to get a complete characterization of when A dominates B via a combinatorial proof. If we can get such a characterization in the general case of Coxeter systems, then applying to the Coxeter systems of type A_n, there is an implied combinatorial proof at least for the flag h-vector inequalities, without having to rely on the Lefschetz Theorem.

Although we are unable to give a complete characterization in this paper, we are able to derive an explicit map for $A \cup B$ to dominate B in the case when A, B are commuting disjoint sets, with B finite (proven as Theorem 4.2.1 in the setting of general Coxeter systems). This map is derived from another result we proved, which states that given a Coxeter system (W, S), if $v \in W$, and $A \subseteq S$ is some subset disjoint from $D_R(v)$, then $w \in W_A$ implies $D_R(vw) \subseteq D_R(v) \cup A$. Also, we prove that for finite Coxeter systems, A dominates B implies $B \subseteq A$ (proven as Theorem 4.2.2), hence generalizing Proposition 5.4 in [NS04], which is the special case of our result for Coxeter systems of type A_n. Our results also gives a partial answer to Problem 2.5 proposed in [Swa06].

In the process of deriving these results, we developed the theory of sequence of braid moves, coining the term "boundary pairs", and we introduced the idea of tagging an element in a reduced expression of a Coxeter group. All of the discussion is made with the aim of applying to descent sets of Coxeter systems. Most of the results in Chapter 3 are new, and all of the results in Chapter 4 are new, which we apply to settle previously unsolved problems. A lot of these results involve the careful study of reduced expressions of words in Coxeter systems, in particular, how the various letters of a reduced expression are changed in a sequence of braid moves, and the ideas involved are purely combinatorial.

In this paper, we have set aside Chapters 1 and 2 to develop the necessary theory needed to explain the results obtained in Chapters 3 and 4. Chapter 5 is an exposition on the applications of our results to the recent work that was briefly discussed above. As a side, we also give a treatment of how inversion tables can be applied to Coxeter systems in Chapter 1.3, and we derive an explicit formula of a reduced expression for all words in Coxeter systems of type A_n.

For notations, we adopt the notations used in [BB05] as far as possible. In particular, for any $n \in \mathbb{Z}^+$, $[n]$ denotes the set of positive integers $\{1, \ldots, n\}$. Each result (proposition, theorem, corollary, lemma) is numbered consecutively within the sections. The symbol $\square$ denotes the end of a proof of a result.

Contents

Chapter 1

Preliminaries

The notion of Coxeter systems was first introduced around 1960 by Jacques Tits as an abstraction of finite reflection groups in geometry. Humphreys gives a good discussion in [Hum92] on how the theory of Coxeter groups can be motivated from the theory of reflection groups, from an algebraic and geometric perspective. In this paper, we give a combinatorial perspective, and much of the basic properties of Coxeter systems we will discuss follow closely the treatment of the combinatorics of Coxeter groups given in [BB05].

In this chapter, we present the basic notations and combinatorial properties of Coxeter groups. In particular, we will introduce the notion of inversion tables and descent sets, and relate the combinatorics of inversion tables and descent sets to the combinatorial properties of Coxeter groups.

1.1 Coxeter Systems

Definition. Let S be a set. A matrix $M : S \times S \to \mathbb{Z}^+ \cup \{\infty\}$, with $m(s, s')$ denoting the (s, s')-th entry of M, is a *Coxeter matrix* if M is a symmetric matrix satisfying

$$m(s, s') = 1 \Leftrightarrow s = s'. \tag{1.1}$$

This matrix M can be represented by a *Coxeter diagram*, which is a graph with vertex set S, and whose edges are the unordered pairs $\{s, s'\}$ satisfying $m(s, s') \geq 3$. By convention, if $m(s, s') \geq 4$, we label the edge $\{s, s'\}$ by $m(s, s')$. The *Coxeter group* of type M is the group $W(M)$ given by the presentation

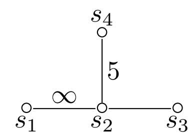
$$W(M) = \langle S \mid (ss')^{m(s,s')} = e, m(s, s') \neq \infty \rangle, \tag{1.2}$$

where e denotes the identity element of $W(M)$. For brevity, we write W instead of $W(M)$, and it is tacitly understood that W corresponds to a Coxeter matrix M. The pair (W, S) is called a *Coxeter system* of type M. S is the set of *Coxeter generators* of

(W, S), or more briefly, the set of generators for W. The cardinality of S is called the *rank* of (W, S). The system is *irreducible* if its Coxeter diagram is connected. Also, if W is finite, then we say the Coxeter system (W, S) is *finite*.

Example 1.1.1. For the set $S = \{s_1, s_2, s_3, s_4\}$, we have the following correspondence between a possible Coxeter matrix with its Coxeter diagram:

$$\begin{bmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 3 & 5 \\ 2 & 3 & 1 & 2 \\ 2 & 5 & 2 & 1 \end{bmatrix} \qquad \longleftrightarrow$$



It is obvious from the definition that up to isomorphism, there is a one-to-one correspondence between Coxeter matrices and Coxeter diagrams. Although it is also true that up to isomorphism, there is a one-to-one correspondence between Coxeter matrices and Coxeter systems, this is not immediately obvious. For a proof, see the remark after Theorem 4.1.3 in [BB05].

In view of these correspondences, there is no ambiguity when referring to *the* corresponding Coxeter matrix and *the* corresponding Coxeter diagram of a Coxeter system (W, S), so the above definition of irreducible Coxeter systems makes sense. In particular, for a given fixed Coxeter matrix M, any two Coxeter systems of type M are necessarily isomorphic, so it makes sense to talk about *the* Coxeter system (W, S) of type M. This implies any two Coxeter groups of type M are also isomorphic, so it also makes sense to talk about *the* Coxeter group of type M. However, we add a word of caution that for any two isomorphic Coxeter groups, it is not necessarily true that they correspond to isomorphic Coxeter systems. See [BB05] for more details.

The notion of 'type' for a Coxeter system suggests there are different types of Coxeter systems. Indeed, all the information about a Coxeter system can be derived from its corresponding Coxeter matrix, and this information is encoded in the Coxeter diagram, so we can classify Coxeter systems according to the structure of their corresponding Coxeter diagrams.

One important class of examples are Coxeter systems of type A_n ($n \in \mathbb{Z}^+$), where A_n denotes the $n \times n$ matrix whose diagonal entries are all 1, whose entries adjacent to the diagonal entries are all 3, and whose other entries are all 2. More explicitly, for every $i, j \in [n]$, the (i, j)-th entry of A_n is given by

$$A_n(i, j) = \begin{cases} 1, & \text{if } i = j \\ 3, & \text{if } |i - j| = 1 \\ 2, & \text{otherwise} \end{cases} .$$

Proposition 1.1.2. The symmetric group S_{n+1} of degree $n+1$ is *the* Coxeter group of type A_n.

Proof: For each $i \in [n]$, let s_i be the transposition $(i, i+1)$ in S_{n+1}. We easily check that $S = \{s_1, \ldots, s_n\}$ is a set of generators for S_{n+1}, such that every pair $(s_i, s_j) \in S \times S$ satisfies $(s_i s_j)^{A_n(i,j)} = \mathrm{Id}$, the identity permutation in S_{n+1}, so (S_{n+1}, S) is a Coxeter system of type A_n. Consequently, by the remark before Example **??**,

(S_{n+1}, S) is *the* Coxeter system of type A_n, and in particular, S_{n+1} is *the* Coxeter group of type A_n. For an alternative proof, see (Proposition 1.5.4, [BB05]). $\square$

We remark that the notation of 'type A_n' used in referring to the symmetric group S_{n+1} is standard in the literature of Coxeter groups. In fact, there is a complete classification of all finite irreducible Coxeter systems, and a classification table can be found in Appendix A. We shall henceforth adopt the standard notation in the classification table when referring to finite irreducible Coxeter systems. For a proof of this classification theorem, see (Chapters 2, 6 in [Hum92]).

Definition. Let $n \in \mathbb{Z}^+$ be given. For each $k \in [n]$, denote s_k as the transposition $(k, k+1)$ in the symmetric group S_{n+1}, and let $S = \{s_1, \dots, s_n\}$. Note that (S_{n+1}, S) is a Coxeter system by Proposition 1.1.2. We shall then denote (S_{n+1}, S) as *the standard Coxeter system of* S_{n+1}.

For the rest of the paper, we denote (W, S) as a Coxeter system (not necessarily finite) with corresponding Coxeter matrix $M = (m(s, s'))_{s,s' \in S}$, unless otherwise stated. The case when $|S| = 1$ is trivial and uninteresting, so we shall assume $|S| \geq 2$. In particular, we allow for S to be infinite. Note that (1.1) implies $s^2 = e$ for all $s \in S$, so in the case when $m(s, s') \neq \infty$, the relation $(ss')^{m(s,s')} = e$ in the presentation (1.2) is equivalent to

$$\underbrace{ss'ss's\cdots}_{m(s,s')} = \underbrace{s'ss'ss'\cdots}_{m(s',s)}. \tag{1.3}$$

In particular, the generators s and s' commute if and only if $m(s, s') = 2$, or equivalently, if and only if s, s' are distinct non-adjacent vertices in the corresponding Coxeter diagram.

Definition. The elements of the Coxeter group W are called *words*. The generators of W (i.e. elements in S) are also called *letters*, and we shall use the terms 'letters' and 'generators' interchangeably. Denote the set $T = \{wsw^{-1} : s \in S, w \in W\}$. The elements of T are called *reflections*. Also, the elements of $S \subseteq T$ are called *simple reflections*.

Definition. Every word $w \in W$ can be written as a finite product of generators $w = s_1 s_2 \cdots s_k$, where $s_i \in S$ are not necessarily distinct. This finite product $s_1 s_2 \cdots s_k$ is called an *expression* for w. For any given expression w_i for the word w, we say the expression has *expression length* k (denoted by $\widetilde{\ell}(w_i) = k$) if there are k (not necessarily distinct) letters appearing in the expression. In particular, the expression $s_1 s_2 \cdots s_k$ has expression length k. If $w_i = s_1 s_2 \cdots s_k$ is an expression for w such that $\widetilde{\ell}(w_i)$ is minimized among all possible expressions w_i for w, then $\widetilde{\ell}(w_i) = k$ is called the *length* of w (denoted by $\ell(w) = k$), and the expression $s_1 s_2 \cdots s_k$ is called a *reduced expression* (or *reduced decomposition* or *reduced word*) for w. Alternatively, we say $s_1 s_2 \cdots s_k$ is *reduced*. By default, the empty product (i.e. $k = 0$) is necessarily reduced, and it refers to the identity element e, with length $\ell(e) = 0$. We shall denote $\mathcal{R}(w)$ as the set of all reduced expressions of w.

Let $F(S)$ denote the free group generated by S, and let $i : S \to F(S)$ be the natural inclusion map. From (1.2), we get $W \cong F(S)/N$, where N is the normal subgroup of $F(S)$ generated by $\{(ss')^{m(s,s')} : m(s, s') \neq \infty\}$. Let $q : F(S) \to W$ be the natural quotient map. At this stage, it is appropriate to remark that there is a distinction between words and expressions. An expression $s_1 \cdots s_k$ for the word w refers to an element $s_1 \cdots s_k$ in $F(S)$ such that $q(s_1 \cdots s_k) = w$. Strictly speaking, a word w is an equivalence class of expressions. We shall reserve the usage of u, v, w to represent words in W, and in cases where the specific choice of expression does not matter, there is no confusion of referring u, v, w as both words and expressions interchangeably. In particular, for $s_1, \dots, s_k \in S$, $ws_1 \cdots s_k$ can be referred to either as a word, or an expression, depending on the context given.

However, to avoid any possible confusion, when we attach subscripts u_i, v_i, w_i, we shall always mean that u_i, v_i, w_i are specific expressions of the words u, v, w respectively. In particular, for any $u, v \in W$, and any expressions u_i, v_i of u, v respectively (not necessarily reduced), we denote $u_i v_i$ as the expression formed by concatenating the expressions u_i and v_i, and we denote uv as the word represented by the expression $u_i v_i$. This distinction becomes important in Chapters 3 and 4.

Note that the expression length function $\widetilde{\ell}$ depends on the expression given. For example, the expressions $ss, ssss, ssssss$ have expression lengths $2, 4, 6$ respectively. Fortunately, the length of a word $w \in W$ does not depend on the choice of expression, and in our example, $ss, ssss, ssssss$ all represent the same word e, so we have $\ell(ss) = \ell(ssss) = \ell(ssssss) = 0$. One obvious consequence is that for any expression w_i, we always have $\ell(w_i) \leq \widetilde{\ell}(w_i)$. Before we derive some properties of the length function, we prove a useful lemma.

Lemma 1.1.3. The map $\varepsilon_0 : s \mapsto -1$ for all $s \in S$, extends to a group homomorphism $\varepsilon : W \to \{\pm 1\}$.

Proof: By the universal property of free groups and the universal property of quotient groups, there exist unique group homomorphisms $\phi : F(S) \to \{\pm 1\}$ and $\varepsilon : W \to \{\pm 1\}$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\varepsilon_0"'] & F(S) \arrow[r, "q"] \arrow[d, "\phi"] & W \arrow[dl, "\varepsilon"] \\ & \{\pm 1\} & \end{tikzcd}$$

□

Definition. Given a Coxeter system (W, S), the group homomorphism ϵ defined in Lemma 1.1.3 is called the *sign representation* of (W, S).

For any $w \in W$, an immediate consequence of Lemma 1.1.3 is the following identity

$$\varepsilon(w) = (-1)^{\ell(w)}, \tag{1.4}$$

which allows us to derive the following basis properties of the length function.

Proposition 1.1.4. For all $k \in \mathbb{Z}^+, u, v, w \in W, s, s_1, \dots, s_k \in S$, the following hold:

(i) $\ell(uw) \equiv \ell(u) + \ell(w) \pmod 2$,

(ii) $\ell(ws) = \ell(w) \pm 1$,

(iii) $\ell(sw) = \ell(w) \pm 1$,

(iv) $\ell(w^{-1}) = \ell(w)$,

(v) $\ell(w) - k \leq \ell(ws_1 \cdots s_k) \leq \ell(w) + k$,

(vi) $\ell(w) - k \leq \ell(s_1 \cdots s_k w) \leq \ell(w) + k$,

(vii) $|\ell(u) - \ell(v)| \leq \ell(uv) \leq \ell(u) + \ell(v)$.

Proof: By considering $\varepsilon(uw) = \varepsilon(u)\varepsilon(w), \varepsilon(ws) = \varepsilon(w)\varepsilon(s), \varepsilon sw = \varepsilon(s)\varepsilon(w)$, parts (i),(ii) (iii) are direct consequences of (1.4). For part (iv), let $s_{i_1} \cdots s_{i_k} \in \mathcal{R}(w)$. Since $(s_{i_1} \cdots s_{i_k})s_{i_k} \cdots s_{i_1} = e$, we have $s_{i_k} \cdots s_{i_1}$ is by definition an expression for w^{-1}, so $\ell(w^{-1}) \leq k$. Let $s_{j_1} \cdots s_{j_{k'}} \in \mathcal{R}(w^{-1})$, where $k' = \ell(w^{-1}) \leq k$. Since $(s_{j_1} \cdots s_{j_{k'}})s_{j_{k'}} \cdots s_{j_1} = e$, we have $s_{j_{k'}} \cdots s_{j_1}$ is by definition an expression for $(w^{-1})^{-1} = w$, so $k = \ell(w) \leq k'$, and (iv) follows. By an inductive argument, part (ii) easily implies (v). In particular, $\ell(w) - k = \ell(ws_1 \cdots s_k)$ if and only if $\ell(ws_1 \cdots s_t) = \ell(ws_1 \cdots s_{t-1}) - 1$ for all $t \in [k]$, and $\ell(w) + k = \ell(ws_1 \cdots s_k)$ if and only if $\ell(ws_1 \cdots s_t) = \ell(ws_1 \cdots s_{t-1}) + 1$ for all $t \in [k]$. An application of parts (iv) and (v) gives (vi). Finally, to show part (vii), if $\ell(u) \geq \ell(v)$, then (vii) is an application of part (v), with $w = u, s_1 \cdots s_k \in \mathcal{R}(v)$, and if $\ell(u) < \ell(v)$, then (vii) is an application of part (vi), with $w = v, s_1 \cdots s_k \in \mathcal{R}(u)$. □

In particular, following the proof of part (iv) above, we get:

$$s_1 \cdots s_k \in \mathcal{R}(w) \Leftrightarrow s_k \cdots s_1 \in \mathcal{R}(w^{-1}). \tag{1.5}$$

Note also that parts (i) and (iv) above imply $\ell(wsw^{-1}) \equiv 1 \pmod 2$ for all $w \in W, s \in S$, so by the definition of T, all reflections have odd lengths. These basic properties will be used repeatedly in the rest of the paper.

1.2 Exchange Property and Deletion Property

The Exchange Property and the Deletion Property are two fundamental combinatorial properties of Coxeter systems, and in fact characterize all Coxeter systems. In this section, we shall mainly state the relevant results and discuss some of their consequences. Most of the proofs are omitted, and we refer the interested reader to [BB05] for detailed proofs of these results.

We remark that there is a 'stronger' version for the Exchange Property, known as the Strong Exchange Property. Although the Exchange Property is a special case of the Strong Exchange Property, we shall see in Theorem 1.2.4 that they are in fact equivalent characterizations of Coxeter systems.

Theorem 1.2.1. (**Strong Exchange Property**) Let $w \in W$ be a given word, and let $s_1 \cdots s_k$ be an expression (not necessarily reduced) for w. If $\ell(tw) \leq \ell(w)$ for some $t \in T$, then $tw = s_1 \cdots \hat{s_i} \cdots s_k$ for some $i \in [k]$. Similarly, if $\ell(wt') \leq \ell(w)$ for some $t' \in T$, then $wt' = s_1 \cdots \hat{s_j} \cdots s_k$ for some $j \in [k]$.

Proof: $\ell(t)$ is odd by the remark after (1.5), so Proposition 1.1.4 implies $\ell(tw)$ and $\ell(w)$ have different parities, which means $\ell(tw) < \ell(w)$. (Theorem 1.4.3, [BB05]) then gives the first assertion, and applying Proposition 1.1.4 to the first assertion, we get the second assertion. □

Corollary 1.2.2. (**Exchange Property**) Let $w \in W$ be a given word, and let $s_1 \cdots s_k$ be a reduced expression for w. If $\ell(sw) \leq \ell(w)$ for some $s \in S$, then $sw = s_1 \cdots \hat{s_i} \cdots s_k$ for some $i \in [k]$. Similarly, if $\ell(ws') \leq \ell(w)$ for some $s' \in S$, then $ws' = s_1 \cdots \hat{s_j} \cdots s_k$ for some $j \in [k]$.

Proof: This is a special case of Theorem 1.2.1. □

Theorem 1.2.3. (**Deletion Property**) Let $w \in W$ be a given word, and let $s_1 \cdots s_k$ be an expression such that $\ell(w) < k$, then $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_k$ for some distinct $i, j \in [k]$.

Proof: See (Proposition 1.4.7, [BB05]). □

Temporarily dropping the assumption that (W, S) is a Coxeter system, let W be an arbitrary group with identity element e, let S be a set of generators for W such that $s^2 = e$ for all $s \in S$, and denote $T = \{wsw^{-1} : s \in S, w \in W\}$. The notions of *expression*, *length* $\ell(w), w \in W$, and *reduced expression* $s_1 \cdots s_k, s_i \in S$ can be defined analogously as earlier. Under these conditions, we make the following definitions:

Definition. Given any $w \in W$ and any expression $s_1 \cdots s_k$ (not necessarily reduced) for w, if $\ell(tw) \leq \ell(w)$ for any $t \in T$ implies $tw = s_1 \cdots \hat{s_i} \cdots s_k$ for some $i \in [k]$, then we say (W, S) has the *Strong Exchange Property.*

Definition. Given any $w \in W$ and any reduced expression $s_1 \cdots s_k$ for w, if $\ell(sw) \leq \ell(w)$ for any $s \in T$ implies $sw = s_1 \cdots \hat{s_i} \cdots s_k$ for some $i \in [k]$, then we say (W, S) has the *Exchange Property.*

Definition. Given any $w \in W$ and any expression $s_1 \cdots s_k$ for w, if $\ell(w) < k$ implies $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_k$ for some distinct $i, j \in [k]$, then we say (W, S) has the *Deletion Property.*

Theorem 1.2.4. Let W be a group with identity element e, and let S be a set of generators for W such that $s^2 = e$ for all $s \in S$. Then the following are equivalent:

(i) (W, S) is a Coxeter system.

(ii) (W, S) has the Strong Exchange Property.

(iii) (W, S) has the Exchange Property.

(iv) (W, S) has the Deletion Property.

Proof: (i) ⇒ (ii) follows from Theorem 1.2.1. (ii) ⇒ (iii) is obvious. The cases (iii) ⇒ (i), (iii) ⇒ (iv), (iv) ⇒ (iii) are proven in (Theorem 1.5.1, [BB05]). □

We remark that for our definition of the Exchange Property and the Strong Exchange Property in the general setting of arbitrary groups, although we have chosen left multiplication of t and s in the respective conditions $\ell(tw) \leq \ell(w)$ and $\ell(sw) \leq \ell(w)$, we could well have chosen right multiplication instead. The choice does not matter, since the above theorem tells us any pair (W, S) that has the Strong Exchange Property or the Exchange Property must necessarily be a Coxeter system, so Theorem 1.2.1 and Corollary 1.2.2 above give the corresponding results for both left and right multiplication.

Having characterized Coxeter systems, we now return to the assumption that (W, S) denotes a Coxeter system. Next, we shall give a few useful consequences of the above properties.

Corollary 1.2.5. Let $w \in W$, let $s_1 \cdots s_k \in \mathcal{R}(w)$, and let $t \in T$. Then the following are equivalent:

(i) $\ell(tw) \leq \ell(w)$.

(ii) $\ell(tw) < \ell(w)$.

(iii) $tw = s_1 \cdots \hat{s}_i \cdots s_k$, for some $i \in [k]$.

(iv) $t = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$, for some $i \in [k]$.

Furthermore, for each reduced expression $s_1 \cdots s_k$ for w and each $t \in T$, the index i in (iii) and (iv) is uniquely determined. Similarly, for $t' \in T$, the following are equivalent:

(i′) $\ell(wt') \leq \ell(w)$.

(ii′) $\ell(wt') < \ell(w)$.

(iii′) $wt' = s_1 \cdots \hat{s}_j \cdots s_k$, for some $j \in [k]$.

(iv′) $t' = s_k \cdots s_{j+1} s_j s_{j+1} \cdots s_k$, for some $j \in [k]$.

Also, for each reduced expression $s_1 \cdots s_k$ for w and each $t' \in T$, the index j in (iii′) and (iv′) is uniquely determined.

Proof: The equivalence (i) $\Leftrightarrow$ (ii) follows from the fact that $\ell(t)$ is odd, which by Proposition 1.1.4 implies $\ell(tw)$ and $\ell(w)$ have different parities, and so cannot be equal. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are proved in (Corollary 1.4.4, [BB05]). As for the set of equivalences (i′) $\Leftrightarrow$ (ii′) $\Leftrightarrow$ (iii′) $\Leftrightarrow$ (iv′), it is an easy consequence of the set of equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) by using the fact that $\ell(w) = \ell(w^{-1})$ for all $w \in W$. □

Corollary 1.2.6. Given any word $w \in W$, the following hold:

(i) Any expression w_i for w contains a subexpression that is a reduced expression for w, obtainable by deleting an even number of letters.

(ii) For any $w_1, w_2 \in \mathcal{R}(w)$, the set of letters appearing in the expression w_1 equals the set of letters appearing in the expression w_2.

(iii) S is a minimal generating set for W, i.e. no Coxeter generator can be expressed in terms of the others.

(iv) Any two expressions for w must have expression lengths of the same parity.

Proof: Parts (i)-(iii) are proven in (Corollary 1.4.8, [BB05]). Part (iv) easily follows from part (i), since part (i) implies any two expressions for w must have expression lengths of the same parity as $\ell(w)$. $\square$

Definition. For any $w \in W$, denote $\S(w) \subseteq S$ as the set of letters appearing in any reduced expression for w. In particular, $\S(e) = \emptyset$. Note that $\S(w)$ is well-defined by Corollary 1.2.6(ii).

Corollary 1.2.7. If $s_1, \ldots, s_n$ are distinct elements in S for some $n \in \mathbb{Z}^+$, then the expression $s_1 s_2 \cdots s_n$ is reduced.

Proof: Suppose $s_1 \cdots s_n$ is not reduced, then by the Deletion Property, there exists distinct $i, j \in [n]$ such that $s_1 \cdots s_n = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_n$. Multiply both sides of the equation on the left by $s_i s_{i-1} \cdots s_1$ and on the right by $s_n s_{n-1} \cdots s_{i+1}$, the identity $s^2 = e$ for all $s \in S$ gives us $s_i = s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1}$, which then contradicts Corollary 1.2.6 part (iii). The assertion then follows. $\square$

1.3 Inversion Tables and Descent Sets

One of the main themes of enumerative combinatorics is the study of permutations of sets, which is well-understood and has found applications in diverse areas in mathematics. Permutations of finite sets can be treated as elements in the symmetric group S_n, and recall from Proposition 1.1.2 that S_n $(n \geq 2)$ is the Coxeter group of type A_{n-1}. Hence, a natural question that arises is: What properties associated to the symmetric group and the permutation of sets can be extended analogously to the general Coxeter group? It is with this motivation that we study the descent sets associated to a Coxeter group, which can be regarded as extensions of the descent sets studied in S_n into the realm of Coxeter systems.

As discussed in (Chapter 1.3, [Sta02]), two of the fundamental statistics associated with a permutation $\pi \in S_n$ are its inversion table and its descent set.

Definition. Let π be a permutation in S_n. The pair $(i, j) \in [n] \times [n]$ is called an *inversion* of π if $i < j$ and $\pi(i) > \pi(j)$. If $i \in [n-1]$ such that $(i, i+1)$ is an inversion of π, then the index i is called a *descent* of π. The *inversion set* of π, denoted by $\mathrm{Inv}(\pi)$, is the set of inversions of π, and the *descent set* of π, denoted by $D(\pi)$, is the set of descents of π. More explicitly, we have

$$\begin{aligned} \mathrm{Inv}(\pi) &= \{(i, j) : i < j, \pi(i) > \pi(j)\} && (1.6) \\ D(\pi) &= \{i : \pi(i) > \pi(i+1)\} && (1.7) \end{aligned}$$

Define the *inversion number* $\mathrm{inv}(\pi)$ of π as $\mathrm{inv}(\pi) = |\mathrm{Inv}(\pi)|$, and define the *descent number* $d(\pi)$ of π as $d(\pi) = |D(\pi)|$. Also, for each $k \in [n]$, denote $b_k = |\{i \in [n] :$

$i < \pi^{-1}(k), \pi(i) > k\}|$. (Note that the inverse map π^{-1} is well-defined, since π is a bijection of $[n]$ onto itself.) In other words, if we denote $a_i = \pi(i)$ for each $i \in [n]$, then $k = a_{\pi^{-1}(k)}$, and b_k counts the number of terms in the sequence $(a_1, \ldots, a_n)$ to the left of $a_{\pi^{-1}(k)}$ that are larger than k. The sequence $(b_1, \ldots, b_n)$ is called the *inversion table* of π. In particular, note that b_n is necessarily 0.

Definition. For any $n \in \mathbb{Z}^+$ and any $\pi \in S_n$, π acts as a permutation on $[n]$. Denote $\pi(i)$ as a_i for each $i \in [n]$. We have $a_1, a_2, \ldots, a_n$ is a permutation of $1, 2, \ldots, n$, and π is uniquely determined by the images $a_1, \ldots, a_n$ under this group action. We then say $a_1 a_2 \cdots a_n$ is a *permutation representation* of π.

Example 1.3.1. Consider $\pi \in S_6$ with permutation representation 362154. The inversion set of π is $\mathrm{Inv}(\pi) = \{(1,3), (1,4), (2,3), (2,4), (2,5), (2,6), (3,4), (5,6)\}$, and the descent set of π is $D(\pi) = \{2, 3, 5\}$. The inversion number and descent number of π are $\mathrm{inv}(\pi) = 8$ and $d(\pi) = 3$. The inversion table of π is $(3, 2, 0, 2, 1, 0)$.

We remark that there is a natural bijection between permutations and inversion tables (see Proposition 1.3.9 in [Sta02]). Also, if $(b_1, \ldots, b_n)$ is the inversion table of a permutation $\pi \in S_n$, then $b_1 + \ldots + b_n$ counts the number of inversions of π, and we get

$$\mathrm{inv}(\pi) = b_1 + \ldots + b_n. \tag{1.8}$$

We are now ready to relate the properties of permutations discussed above to the setting of Coxeter systems.

Lemma 1.3.2. Let (S_n, S) $(n \geq 2)$ be the standard Coxeter system of the symmetric group S_n. Then for any $w \in S_n$, we have $\ell(w) = \mathrm{inv}(w)$.

Proof: See (Proposition 1.5.2, [BB05]) □

In fact, if we know the inversion table of w, we can say even more. First, we define the notions of ascending expressions and descending expressions.

Definition. Let (W, S) be a Coxeter system of finite rank n, and label the elements in S as $s_1, \ldots, s_n$. Let $t_1, t_2 \in [n]$. If $t_1 \leq t_2$, denote $\acute{\beta}(t_1 : t_2)$ as the expression $s_{t_1} s_{t_1+1} \cdots s_{t_2}$ and denote $\grave{\beta}(t_2 : t_1)$ as the expression $s_{t_2} s_{t_2-1} \cdots s_{t_1}$, and if $t_1 > t_2$, set each of $\acute{\beta}(t_1 : t_2), \grave{\beta}(t_2 : t_1)$ as the empty expression. Note that if $t_1 \leq t_2$, then $\acute{\beta}(t_1 : t_2)$ and $\grave{\beta}(t_2 : t_1)$ each has expression length $t_2 - t_1 + 1$, while if $t_1 > t_2$, then $\acute{\beta}(t_1 : t_2)$ and $\grave{\beta}(t_2 : t_1)$ each has expression length 0. We call $\acute{\beta}(t_1 : t_2)$ an *ascending expression*, and we call $\grave{\beta}(t_1 : t_2)$ a *descending expression*.

Next, we record some obvious observations:

Proposition 1.3.3. Let (S_n, S) (with $n \geq 2$) be the standard Coxeter system of the symmetric group S_n. For any $w \in S_n$, let $a_1 \cdots a_n$ be the permutation representation of w. Then for any $j \in [n-1]$, the permutation representation of ws_j is $a_1 \cdots a_{j-1} a_{j+1} a_j a_{j+2} \cdots a_n$, obtained from $a_1 \cdots a_n$ by swapping the terms a_j and a_{j+1}.

Proof: This is obvious, since s_j is just the transposition $(j, j+1)$. □

Proposition 1.3.4. Let (S_n, S) (with $n \geq 2$) be the standard Coxeter system of the symmetric group S_n. For any $w \in S_n$, let $a_1 \cdots a_n$ be the permutation representation of w. Then for any $j \in [n-1]$, the permutation representation of $ws_1s_2\cdots s_j$ is $a_2a_3\cdots a_{j+1}a_1a_{j+2}\cdots a_n$, obtained from $a_1\cdots a_n$ by shifting the term a_1 from the left of a_2 to in between terms a_{j+1} and a_{j+2}.

Proof: This is an application of Proposition 1.3.3 and is obvious. $\square$

Now, given the inversion table of $w \in S_n$, we shall find an explicit reduced expression for w.

Theorem 1.3.5. Let (S_n, S) (with $n \geq 2$) be the standard Coxeter system of the symmetric group S_n. Let $w \in S_n$, and let $(b_1, \ldots, b_n)$ be the inversion table of w. For each $k \in [n-1]$, denote v_k as the expression $\acute{\beta}(k : k-1+b_k)$. Then the expression $v_{n-1}\cdots v_2v_1$ is a reduced expression for w.

Proof: We shall prove this by induction on n. The base case $n = 2$ is trivially true. Suppose that for some integer $N > 2$, the assertion is true for all integers n satisfying $2 \leq n < N$. Consider the case $n = N$, choose a word $w \in S_N$, let $a_1\cdots a_N$ be its permutation representation, and let $(b_1, \ldots, b_N)$ be the inversion table of w. By definition, $a_1, \ldots, a_N$ is just a permutation of $1, \ldots, N$. Let $j \in [N]$ be the unique index such that $a_j = 1$. Let $v \in S_N$ be the permutation given by the permutation representation $1a_1\cdots a_{j-1}a_{j+1}\cdots a_N$, and in particular, v fixes 1. From Proposition 1.3.4, we get $w = vs_1\cdots s_{j-1}$. Since 1 is the smallest integer in $[N]$, all $j-1$ terms to the left of a_j in $a_1\cdots a_N$ are larger that $a_j = 1$, so $b_1 = j-1$ by definition, hence $w = vs_1\cdots s_{j-1}$ is equivalent to $w = vv_1$.

For each $i \in [N-1]$, denote a'_i as $a_i - 1$ if $i < j$, and denote a'_i as $a_{i+1} - 1$ if $i \geq j$. In other words, the sequence $a'_1, \ldots, a'_{N-1}$ is obtained from $a_1, \ldots, a_N$ by subtracting 1 from each term, and then omitting the term 0. Note that $a'_1, \ldots, a'_{N-1}$ is a permutation of $1, \ldots, N-1$, and denote v' as the unique permutation in S_{N-1} with permutation representation $a'_1\cdots a'_{N-1}$. Denote $(b'_1, \ldots, b'_{N-1})$ as the inversion table of w'. By the construction of w', we easily see that $b'_i = b_{i+1}$ for each $i \in [N-1]$.

Now, since v fixes 1 and permutes the integers $2, \ldots, N$, we can treat v as a permutation on $N-1$ elements. Let G be the subgroup of S_N such that every permutation in G fixes 1, and let $S' = \{s_2, \ldots, s_N\}$. Observe that $G \cong S_{N-1}$ and that S' is a set of generators for G, so that (G, S') and (S_{N-1}, S) are isomorphic as Coxeter systems. Under this isomorphism, v corresponds to v', so by applying the induction hypothesis on v' and using this isomorphism, we get

$$\begin{aligned} v &= \acute{\beta}(N-1 : N-1+b'_{N-2})\acute{\beta}(N-2 : N-2+b'_{N-3})\cdots\acute{\beta}(2 : 1+b'_1) \\ &= \acute{\beta}(N-1 : N-1+b_{N-1})\acute{\beta}(N-2 : N-2+b_{N-2})\cdots\acute{\beta}(2 : 1+b_2) \\ &= v_{N-1}v_{N-2}\cdots v_2. \end{aligned}$$

Consequently, since $w = vv_1$, we get $w = v_{N-1}v_{N-2}\cdots v_1$, which by definition has expression length $b_1 + \ldots + b_{N-1}$. Finally, from (1.8) and Lemma 1.3.2, since $b_N = 0$, we have $\ell(w) = \mathrm{inv}(w) = b_1 + \ldots + b_{N-1}$, therefore this expression $v_{N-1}v_{N-2}\cdots v_1$ for w is reduced, and by induction, the assertion follows. $\square$

Example 1.3.6. Recall from Example 1.3.1 that the permutation $\pi \in S_6$ represented by 362154 has inversion table $(3, 2, 0, 2, 1, 0)$. Theorem 1.3.5 then says

$$\acute{\beta}(5, 4+1)\acute{\beta}(4, 3+2)\acute{\beta}(3, 2+0)\acute{\beta}(2, 1+2)\acute{\beta}(1, 0+3) = s_5s_4s_5s_2s_3s_1s_2s_3$$

is a reduced expression for $\pi \in S_6$.

Remark. Consider the standard Coxeter system (S_n, S) for S_n. For any $w \in S_n$, it follows from Theorem 1.3.5 that if we know the inversion table of w, then we can explicitly construct a reduced expression for w. A natural question that follows is whether the notion of inversion table can be extended analogously to the general Coxeter system. If such an extension is possible, then we should be able to explicitly construct a reduced expression for any given word in any Coxeter system.

The discussion of the relation between inversion tables and Coxeter systems so far has been fruitful. Next, we shift our attention to the other statistic of permutations of sets - descent sets. Again, let (S_n, S) be the standard Coxeter system for S_n. For any word $w \in S_n$, let $a_1 \cdots a_n$ be the permutation representation of w. From Proposition 1.3.3, we get the following:

$$\operatorname{inv}(ws_i) = \begin{cases} \operatorname{inv}(w) + 1, & \text{if } w(i) < w(i+1) \\ \operatorname{inv}(w) - 1, & \text{if } w(i) > w(i+1) \end{cases}. \tag{1.9}$$

Applying Lemma 1.3.2, this is equivalent to

$$\ell(ws_i) = \begin{cases} \ell(w) + 1, & \text{if } w(i) < w(i+1) \\ \ell(w) - 1, & \text{if } w(i) > w(i+1) \end{cases}. \tag{1.10}$$

We now consider the following definitions:

Definition. For any $w \in W$, denote

$$D_L(w) = \{s \in S : \ell(sw) < \ell(w)\}, \quad D_R(w) = \{s \in S : \ell(ws) < \ell(w)\},$$
$$T_L(w) = \{t \in T : \ell(tw) < \ell(w)\}, \quad T_R(w) = \{t \in T : \ell(wt) < \ell(w)\}.$$

$D_L(w)$ is called the *left descent set* of w, while $D_R(w)$ is called the *right descent set* of w. $T_L(w)$ is called the set of *left associated reflections* to w, while $T_R(w)$ is called the set of *right associated reflections* to w. The subscripts 'L' and 'R' are mnemonic for 'left' and 'right' respectively.

For any permutation $w \in S_n$, it follows from (1.10) that $i \in D(w)$ if and only if $s_i \in D_R(w)$. As discussed in [BB05], this is the reason why $D_R(w)$ are known as descent sets. The following results give the relation between descent sets and reduced expressions of words.

Lemma 1.3.7. For all $w \in W$ and $s \in S$, the following hold:

(i) $s \in D_L(w)$ if and only if some reduced expression for w begins with the letter s.

(ii) $s \in D_R(w)$ if and only if some reduced expression for w ends with the letter s.

Proof: See (Corollary 1.4.6, [BB05]). □

Proposition 1.3.8. For all $w \in W$, we have $T_R(w) = T_L(w^{-1})$ and $D_R(w) = D_L(w^{-1})$.

Proof: This is an immediate consequence of Proposition 1.1.4 (iv). □

Chapter 2

Poset Structure, Parabolic Subgroups and Quotients

In this chapter, we shall discuss two partial order relations on Coxeter systems - the Bruhat order, and the weak order. As an overview, the Bruhat order is defined by reflections (i.e. elements in T), while the weak order is defined by simple reflections (i.e. elements in $S \subseteq T$), so weak order necessarily implies Bruhat order, but not conversely. In this sense, the weak order is 'weaker' than the Bruhat order, hence its name.

We shall first explore some basic properties of the Bruhat order and the weak order. Next, we shall introduce the parabolic subgroups and quotient groups of Coxeter groups, and discuss properties of unique factorization in the setting of Coxeter systems. Finally, from a combinatorial perspective, we explore the relations between the largest elements of the Coxeter group and its corresponding parabolic and quotient subgroups.

2.1 Bruhat Order

The Bruhat order of a Coxeter system (W, S) is determined by its set of reflections T, where we recall from Chapter 1.1 that $T = \{wsw^{-1} : s \in S, w \in W\}$. The notion of 'reflections' suggests a geometric interpretation, and indeed, the Bruhat order was first considered in the 1930s with the purpose of describing the containment ordering of Schubert varieties in flag manifolds, Grassmannians, and other homogenous spaces. Since then, the Bruhat order has found various applications in geometry and representation theory. Although such applications are interesting, they are not used in the discussion of later chapters, so we shall deviate from the conventional geometric approach and deal only with the relevant combinatorial properties of the Bruhat order. The interested reader is referred to [Hum92] for a detailed discussion of reflection groups.

Definition. Let $u, w \in W$. Then

(i) For a given $t \in T$, denote $u \xrightarrow{t} w$ to mean that $ut = w$ and $\ell(u) < \ell(w)$.

(ii) Denote $u \to w$ to mean that $u \xrightarrow{t} w$ for some $t \in T$.

(iii) Denote $u \leq w$ to mean there exist $k \in \mathbb{Z}_{\geq 0}$ and $u_0, \ldots, u_k \in W$ such that

$$u = u_0 \to u_1 \to \cdots \to u_k = w.$$

The *Bruhat graph* is the directed graph whose nodes are the elements of W, and whose edges are given by (ii). The *Bruhat order* is the partial order relation defined on the set W, given by part (iii).

First, we record some obvious observations that follow immediately from the definition:

Lemma 2.1.1. The following are obvious:

(i) For any $u, w \in W$, $u < w$ implies $\ell(u) < \ell(w)$.

(ii) For all $u \in W$ and all $t \in T$, we have $u < ut$ if and only if $\ell(u) < \ell(ut)$.

(iii) The identity element e satisfies $e \leq w$ for all $w \in W$. In particular, if $s_1 \cdots s_k \in \mathcal{R}(w)$, then we get the induced chain $e \to s_1 \to s_1 s_2 \to \cdots \to s_1 \cdots s_k = w$.

Next, we shall list a few relevant results related to Bruhat order. Of great importance is the Subword Property (Theorem 2.1.3) and the Chain Property (Theorem 2.1.6). The proofs of all these results can be found in [BB05], and the reader is referred to the corresponding relevant sections.

Lemma 2.1.2. For distinct $u, w \in W$, let $s_1 \cdots s_k \in \mathcal{R}(w)$, and suppose that some reduced expression for u is a sub-expression of $s_1 \cdots s_k$. Then there exists $v \in W$ such that the following hold:

(i) $v > u$.

(ii) $\ell(v) = \ell(u) + 1$.

(iii) Some reduced expression for v is a sub-expression of $s_1 \cdots s_k$.

Proof: See (Lemma 2.2.1, [BB05]). □

Theorem 2.1.3. (**Subword Property**) Let $u, w \in W$, and let $w_i = s_1 \cdots s_k \in \mathcal{R}(w)$. Then $u \leq w$ if and only if there exists a sub-expression w_i' of w_i such that $w_i' \in \mathcal{R}(u)$. In other words,

$$u \leq w \Leftrightarrow u = s_{i_1} s_{i_2} \cdots s_{i_t} \text{ is reduced for some } 1 \leq i_1 < i_2 < \ldots < i_t \leq k.$$

Proof: See (Theorem 2.2.2, [BB05]). □

Corollary 2.1.4. For $u, w \in W$, the following are equivalent:

(i) $u \leq w$.

(ii) Every reduced expression for w has a sub-expression that is a reduced expression for u.

(iii) Some reduced expression for w has a sub-expression that is a reduced expression for u.

Proof: See (Corollary 2.2.3, [BB05]). $\square$

Corollary 2.1.5. The mapping $w \mapsto w^{-1}$ is an automorphism of Bruhat order. In other words, for $u, w \in W$, we have $u \leq w$ if and only if $u^{-1} \leq w^{-1}$.

Proof: See (Corollary 2.2.5, [BB05]). $\square$

Theorem 2.1.6. (**Chain Property**) If $u, w \in W$ such that $u < w$, then there exists a chain $u = u_0 < u_1 < \ldots < u_k = w$ such that $\ell(u_i) = \ell(u) + i$ for every $i \in [k]$.

Proof: This immediately follows from Lemma 2.1.2 and the Subword Property. $\square$

Definition. We shall use the notation "$u \lhd v$" or "$v \rhd u$" to mean a covering in Bruhat order. Thus, by the Chain Property, $u \lhd v$ means that $u < v$ and $\ell(u) + 1 = \ell(v)$. Similarly, $v \rhd u$ means that $v > u$ and $\ell(v) = \ell(u) + 1$.

In particular, the Chain Property shows that Bruhat order is a graded poset whose rank function is the length function ℓ. This is also true for any Bruhat interval $[u, v]$.

2.2 Weak Order

In this section, we shall explore the weak order of Coxeter groups.

Definition. Let $u, w \in W$. Then

(i) $u \leq_R w$ means that $w = us_1 \cdots s_k$ for some $k \in \mathbb{Z}_{\geq 0}$ and some $s_1, \ldots, s_k \in S$, such that $\ell(us_1 \cdots s_i) = \ell(u) + i$ for every $i \in [k]$.

(ii) $u \leq_L w$ means that $w = s_k s_{k-1} \cdots s_1 u$ for some $k \in \mathbb{Z}_{\geq 0}$ and some $s_1, \ldots, s_k \in S$, such that $\ell(s_i \cdots s_1 u) = \ell(u) + i$ for every $i \in [k]$.

The partial order relations $\leq_R$ and $\leq_L$ are called the *right weak order* and the *left weak order* respectively.

Although the right and left weak order are distinct partial orderings of W, they are isomorphic via the map $w \mapsto w^{-1}$. For any $u, w \in W$, one important relation between the weak order and the Bruhat order is the following:

$$u \leq_R w \quad \text{or} \quad u \leq_L w \Rightarrow u \leq w \tag{2.1}$$

Next, we give a list of properties of the weak order:

Proposition 2.2.1. Let $u, w \in W$, then the following hold:

(i) There is a one-to-one correspondence between elements in $\mathcal{R}(w)$ and maximal chains in the interval $[e, w]_R$.

(ii) $u \leq_R w \Leftrightarrow \ell(u) + \ell(u^{-1}w) = \ell(w)$.

(iii) If W is finite, then $w \leq_R w_0$ for all $w \in W$, where w_0 denotes the unique element in W of maximal length.

(iv) (**Prefix Property**) $u \leq_R w$ if and only if there exists $k, m \in \mathbb{Z}_{\geq 0}$, and some $s_1, \ldots, s_k, s'_1, \ldots, s'_m \in S$ such that $s_1 \cdots s_k \in \mathcal{R}(u)$ and $s_1 \cdots s_k s'_1 \cdots s'_m \in \mathcal{R}(w)$.

(v) (**Chain Property**) If $u <_R w$, then there exists a chain $u = u_0 <_R u_1 <_R \ldots <_R u_k = w$ such that $\ell(u_i) = \ell(u) + i$ for every $i \in [k]$.

(vi) W under the weak order is a graded poset ranked by the length function ℓ, and so is every interval $[u, w]_R$.

(vii) If $s \in D_L(u) \cap D_L(w)$, then $u \leq_R w$ if and only if $su \leq_R sw$.

(viii) $u \leq_R w$ if and only if $T_L(u) \subseteq T_L(w)$.

Proof: Parts (i)-(vii) are proven in (Proposition 3.1.2, [BB05]), while part (viii) is proven in (Proposition 3.1.3, [BB05]). $\square$

Proposition 2.2.2. Let $v, w \in W$. Then the following are equivalent:

(i) $v \leq_R vw$.

(ii) $\ell(vw) = \ell(v) + \ell(w)$.

(iii) $v_i w_i$ is reduced for some $v_i \in \mathcal{R}(v), w_i \in \mathcal{R}(w)$.

(iv) $v_i w_i$ is reduced for all $v_i \in \mathcal{R}(v), w_i \in \mathcal{R}(w)$.

Proof: The equivalence (i) $\Leftrightarrow$ (ii) is an immediate consequence of Proposition 2.2.1(ii). The equivalence (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is trivially true by the definition of reduced expressions. $\square$

Proposition 2.2.3. Let $v, w \in W$. If $v \leq_R vw$, then $D_R(v) \cap D_L(w) = \emptyset$.

Proof: We shall prove its contrapositive. If $D_R(v) \cap D_L(w) \neq \emptyset$, then choosing some $s_0 \in D_R(v) \cap D_L(w)$, Lemma 1.3.7 tells us there is some expression $v_i \in \mathcal{R}(v)$ ending in s_0, and there is some expression $w_i \in \mathcal{R}(w)$ beginning with s_0, hence $v_i w_i$ is obviously not reduced, so Proposition 2.2.2 implies $v \not\leq_R vw$. $\square$

Note that the converse of Proposition 2.2.3 is not true. For example, if $s, s' \in S$ are distinct generators satisfying $m(s, s') = 3$, then $D_R(ss') = \{s'\}$ and $D_L(ss') = \{s\}$, and we have $D_R(ss') \cap D_L(ss') = \emptyset$. Yet $ss'ss' = s's$, and we obviously have $ss' \not\leq_R s's$. However, there is still a partial converse as follows:

Remark. Let $v, w \in W$. If $D_R(v) \cap \S(w) = \emptyset$, then $v \leq_R vw$.

At this point, we do not have the necessary tools to prove this partial converse. However, we shall see in the next section that this partial converse easily follows from unique factorization.

2.3 Unique Factorization

In this section, we shall introduce the notions of parabolic subgroups and quotients. One very useful result of studying the parabolic subgroups and quotients of Coxeter systems is that we obtain a unique factorization property of the words in W.

Definition. Let $J \subseteq S$. We denote W_J to be the subgroup of W generated by the set J, and we call W_J the *parabolic subgroup* of W *generated by* J. Also, we denote

$$W^J = \{w \in W : ws > w \text{ for all } s \in J\}. \tag{2.2}$$

$$^J W = \{w \in W : sw > w \text{ for all } s \in J\}. \tag{2.3}$$

We call W^J and $^J W$ *quotients* of W. Also, we denote $\ell_J(\cdot)$ as the length function of W_J with respect to the set of generators J.

Some basic properties of parabolic subgroups are listed below:

Proposition 2.3.1. Let $I, J \subseteq S$. The following hold:

(i) (W_J, J) is a Coxeter system.

(ii) $\ell_J(w) = \ell(w)$ for all $w \in W_J$.

(iii) $W_I \cap W_J = W_{I \cap J}$.

(iv) $\langle W_I \cup W_J \rangle = W_{I \cup J}$.

(v) $W_I = W_J \Rightarrow I = J$.

Proof: See (Proposition 2.4.1, [BB05]). □

Definition. For $I \subseteq J \subseteq S$, define the following:

$$\mathcal{D}_I^J = \{w \in W : I \subseteq D_R(w) \subseteq J\}. \tag{2.4}$$

$${}_I^J\mathcal{D} = \{w \in W : I \subseteq D_L(w) \subseteq J\}. \tag{2.5}$$

Sets of the form $\mathcal{D}_I^J$ are called *right descent classes*, while sets of the form ${}_I^J\mathcal{D}$ are called *left descent classes*.

By the definition of descent classes, it easily follows that we have the following identities

$$W^J = \{w \in W : ws > w \ \forall\ s \in J\} = \{w \in W : D_R(w) \subseteq S \setminus J\} = \mathcal{D}_\emptyset^{S \setminus J}. \tag{2.6}$$

$$^J W = \{w \in W : sw > w \ \forall\ s \in J\} = \{w \in W : D_L(w) \subseteq S \setminus J\} = {}_\emptyset^{S \setminus J}\mathcal{D}. \tag{2.7}$$

This just means that W^J is the set of all words in W whose right descent sets are disjoint from J, while ${}^JW$ is the set of all words in W whose left descent sets are disjoint from J. In fact, in view of Lemma 1.3.7, we easily get the following lemma:

Lemma 2.3.2. Let $J \subseteq S$. An element w is in W^J if and only if no reduced expression for w ends with a letter from J. Similarly, an element w' is in ${}^JW$ if and only if no reduced expression for w' begins with a letter from J.

Proof: This directly follows from Lemma 1.3.7. □

We can also refer to quotients of parabolic subgroups naturally. Letting $I \subseteq J \subseteq S$, we have the following identities:

$$(W_J)^I = \{w \in W_J : ws > w \ \forall\ s \in I\} = \{w \in W_J : D_R(w) \subseteq J \setminus I\} = \mathcal{D}_\emptyset^{J\setminus I}. \quad (2.8)$$

$${}^I(W_J) = \{w \in W_J : sw > w \ \forall\ s \in I\} = \{w \in W_J : D_L(w) \subseteq J \setminus I\} = {}^{J\setminus I}_{\emptyset}\mathcal{D}. \quad (2.9)$$

We know come to the main result of this section:

Theorem 2.3.3. Let $J \subseteq S$. Then every $w \in W$ has a unique factorization

$$w = w^J \cdot w_J \quad (2.10)$$

such that $w^J \in W^J$ and $w_J \in W_J$, where for this factorization, we have

$$\ell(w) = \ell(w^J) + \ell(w_J). \quad (2.11)$$

Similarly, every $v \in W$ has a unique factorization

$$v = v_J \cdot {}^Jv \quad (2.12)$$

such that $v_J \in W_J$ and ${}^Jv \in {}^JW$, where for this factorization, we have

$$\ell(v) = \ell(v_J) + \ell({}^Jv). \quad (2.13)$$

Proof: The first assertion for the unique factorization of w is proven in (Proposition 2.4.4, [BB05]). The second assertion for the unique factorization of v easily follows from the first by observing that ${}^JW = (W^J)^{-1}$ by definition. □

Parabolic subgroups have complete systems of combinatorially distinguished coset representatives, as shown by the following corollary:

Corollary 2.3.4. Let $J \subseteq S$. Then the following hold:

(i) Each left coset wW_J has a unique representative of minimal length. The system of such minimal coset representatives is $W^J = D_\emptyset^{S\setminus J}$.

(ii) Each right coset W_Jw has a unique representative of minimal length. The system of such minimal coset representatives is ${}^JW = {}^{S\setminus J}_{\emptyset}D$.

(iii) If W_J is finite, then each left coset wW_J has a unique representative of maximal length. The system of such maximal coset representatives is $\mathcal{D}_J^S$.

(iv) If W_J is finite, then each right coset $W_J w$ has a unique representative of maximal length. The system of such maximal coset representatives is ${}^S_J\mathcal{D}$.

Proof: See (Corollary 2.4.5, [BB05]) for the proof of (i) and (iii). □

Now, consider the case when (W, S) is a Coxeter system of finite rank (i.e. S is finite). Label the elements in S as $\{s_1, \ldots, s_n\}$, and for each $i \in [n]$, denote $Q_i = (W_{\{s_1,\ldots s_i\}})^{\{s_1,\ldots,s_{i-1}\}}$ if $i > 1$, and for the case $i = 1$, denote $Q_1 = W_{s_1} = \{e, s_1\}$. By repeatedly applying Theorem 2.3.3, we get the following:

Corollary 2.3.5. The product map $Q_1 \times \cdots \times Q_n \to W$, defined by

$$(q_1, q_2, \ldots, q_n) \mapsto q_n q_{n-1} \cdots q_1$$

is a bijection satisfying $\ell(q_n q_{n-1} \cdots q_1) = \ell(q_1) + \ell(q_2) + \ldots + \ell(q_n)$.

Proof: This is just the application of Theorem 2.3.3 inductively on $Q_1, Q_2, \ldots, Q_n$. □

Recall from Proposition 2.2.3 that given any $v, w \in W$, we have $v \leq_R vw$ implies $D_R(v) \cap D_L(w) = \emptyset$. We also showed that its converse is not true by giving a counter-example, and we proposed a partial converse. In particular, note that for any word $w \in W$, we have $\S(w) = A$ for some subset $A \subseteq S$ implies $w \in W_A$. We are now ready to prove that partial converse:

Proposition 2.3.6. Let $A \subseteq S$, let $v \in W$, and let $w \in W_A$. If $D_R(v) \cap A = \emptyset$, then $v \leq_R vw$.

Proof: By definition, we have $v \in W^A$ and $w \in W_A$, hence (2.11) gives $\ell(vw) = \ell(v) + \ell(w)$, so by Proposition 2.2.2(ii), we get $v \leq_R vw$. □

2.4 Largest Elements

For a general Coxeter system (W, S), there may not necessarily be any element having maximal length. For example, if S is infinite, then Corollary 1.2.7 clearly shows that there are elements in W of arbitrarily large length. However, if (W, S) is a finite Coxeter system, then there must exist an element of maximal length. It is not hard to show that this element is unique. (See Proposition 2.2.9, [BB05] for a proof.) We can then make the following definition:

Definition. If (W, S) is a finite Coxeter system, then we denote w_0 as the unique element of maximal length. This notation 'w_0' is standard in the literature of Coxeter systems. We say w_0 is the *largest element* in W.

Proposition 2.4.1. Let (S_n, S) (with $n \geq 2$) be the standard Coxeter system of the symmetric group S_n. Then largest element w_0 in S_n corresponds to the permutation representation $n \cdots 21$, and w_0 has a reduced expression

$$s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n) \cdots (s_1 \cdots s_n). \tag{2.14}$$

Proof: Consider an arbitrary word $w \in S_n$ with permutation representation $a_1 \cdots a_n$ and inversion table $(b_1, \ldots, b_n)$. For each $k \in [n]$, b_k counts the number of terms to the left of $a_{w^{-1}(k)}$ that are larger than k. Since there are $n-k$ integers in $[n]$ larger than k, we must have $b_k \leq n-k$. Let $w' \in S_n$ correspond to permutation representation $n(n-1)\cdots 21$, and note that w' has the inversion table $(n-1, n-2, \ldots, 1, 0)$, so equality holds in $b_k \leq n-k$ for every $k \in [n]$. By Lemma 1.3.2, w' has precisely the largest possible length. So by the uniqueness of w_0, and by applying Theorem 1.3.5, the result follows. □

Next, we give a list of useful results related to w_0:

Proposition 2.4.2. Let (W, S) be a finite Coxeter system. Then w_0 exists, and for all $w \in W$, the following hold:

(i) $w_0^2 = e$.

(ii) $w_0^{-1} = w_0$.

(iii) $\ell(ww_0) = \ell(w_0w) = \ell(w_0) - \ell(w)$.

(iv) $\ell(w_0ww_0) = \ell(w)$.

(v) $\ell(w_0) = |T|$.

Proof: Parts (i) and (iii)-(v) are proven in (Proposition 2.3.2 and Corollary 2.3.3, [BB05]). As for part (ii), substitute $w = w_0^{-1}$ into part (iii) to get $\ell(w_0^{-1}) = \ell(w_0)$, so by the uniqueness of w_0, (ii) follows. □

Proposition 2.4.3. Let (W, S) be any Coxeter system, and let $w \in W$. Then the following are equivalent:

(i) $D_L(w) = S$.

(ii) $D_R(w) = S$.

(iii) W is finite, and $w = w_0$.

Proof: The equivalence (i) $\Leftrightarrow$ (iii) is proven in (Proposition 2.3.1, [BB05]). As for the equivalence (ii) $\Leftrightarrow$ (iii), Proposition 1.3.8 gives us $D_R(w) = S$ if and only if $D_L(w^{-1}) = S$, and the equivalence (i) $\Leftrightarrow$ (iii) gives us $D_L(w^{-1}) = S$ if and only if W is finite and $w^{-1} = w_0$, so by $w_0^{-1} = w_0$ (Proposition 2.4.2 part (ii)), the equivalence (ii) $\Leftrightarrow$ (iii) follows. □

Proposition 2.4.4. For both the Bruhat order and the weak order on a finite Coxeter system, the following hold:

(i) $w \mapsto ww_0$ and $w \mapsto w_0w$ are anti-automorphisms.

(ii) $w \mapsto w_0ww_0$ is an automorphism.

Proof: See (Proposition 2.3.4, [BB05]) and (Proposition 3.1.5, [BB05]). □

Proposition 2.4.5. Let (W, S) be a finite Coxeter system. Then for any $w \in W$, the following hold:

(i) $T_L(ww_0) = T \setminus T_L(w)$ and $T_R(w_0w) = T \setminus T_R(w)$.

(ii) $T_L(w_0w) = w_0(T \setminus T_L(w))w_0 = T \setminus (w_0T_L(w)w_0)$ and $T_R(ww_0) = w_0(T \setminus T_R(w))w_0 = T \setminus (w_0T_R(w)w_0)$.

(iii) $T_L(w_0ww_0) = w_0T_L(w)w_0$ and $T_R(w_0ww_0) = w_0T_R(w)w_0$.

(iv) $D_L(ww_0) = S \setminus D_L(w)$ and $D_R(w_0w) = S \setminus D_R(w)$.

(v) $D_L(w_0w) = w_0(S \setminus D_L(w))w_0 = S \setminus (w_0D_L(w)w_0)$ and $D_R(ww_0) = w_0(S \setminus D_R(w))w_0 = S \setminus (w_0D_R(w)w_0)$.

(vi) $D_L(w_0ww_0) = w_0D_L(w)w_0$ and $D_R(w_0ww_0) = w_0D_R(w)w_0$.

Proof: In view of Proposition 1.3.8 and the fact that $w_0^{-1} = w_0$ (Proposition 2.4.2), replacing w with w^{-1} in the the first statement of each part gives the corresponding second part, thus it suffices to prove only the first statement of every part. From the anti-automorphism $w \mapsto ww_0$ (Proposition 2.4.4), we have $tww_0 < ww_0 \Leftrightarrow tw > w$ for all $t \in T$, and in particular, $sww_0 < ww_0 \Leftrightarrow sw > w$ for all $s \in S$. Similar, the anti-automorphism $w \mapsto w_0w$ gives $tw_0w < w_0w \Leftrightarrow w_0tw_0w > w$ for all $t \in T$ and $sw_0w < w_0w \Leftrightarrow w_0sw_0w > w$ for all $s \in S$, while the automorphism $w \mapsto w_0ww_0$ gives $tw_0ww_0 < w_0ww_0 \Leftrightarrow w_0tw_0w < w$ for all $t \in T$ and $sw_0ww_0 < w_0ww_0 \Leftrightarrow w_0sw_0w < w$ for all $s \in S$. Consequently, we have the following:

$$t \in T_L(ww_0) \Leftrightarrow tww_0 < ww_0 \Leftrightarrow tw > w \Leftrightarrow t \in T \setminus T_L(w).$$
$$t \in T_L(w_0w) \Leftrightarrow tw_0w < w_0w \Leftrightarrow w_0tw_0w > w \Leftrightarrow w_0tw_0 \in T \setminus T_L(w).$$
$$t \in T_L(w_0ww_0) \Leftrightarrow tw_0ww_0 < w_0ww_0 \Leftrightarrow w_0tw_0 < w \Leftrightarrow w_0tw_0 \in T_L(w).$$
$$s \in D_L(ww_0) \Leftrightarrow sww_0 < ww_0 \Leftrightarrow sw > w \Leftrightarrow s \in S \setminus D_L(w).$$
$$s \in D_L(w_0w) \Leftrightarrow sw_0w < w_0w \Leftrightarrow w_0sw_0w > w \Leftrightarrow w_0sw_0 \in S \setminus D_L(w).$$
$$s \in D_L(w_0ww_0) \Leftrightarrow sw_0ww_0 < w_0ww_0 \Leftrightarrow w_0sw_0 < w \Leftrightarrow w_0sw_0 \in D_L(w).$$

Finally, note that

$$w_0tw_0 \in T \setminus T_L(w) \Leftrightarrow t \in w_0(T \setminus T_L(w))w_0 \Leftrightarrow t \in T \setminus (w_0T_L(w)w_0).$$
$$w_0tw_0 \in T_L(w) \Leftrightarrow t \in w_0T_L(w)w_0.$$
$$w_0sw_0 \in S \setminus D_L(w) \Leftrightarrow s \in w_0(S \setminus D_L(w))w_0 \Leftrightarrow s \in S \setminus (w_0D_L(w)w_0).$$
$$w_0sw_0 \in D_L(w) \Leftrightarrow s \in w_0D_L(w)w_0.$$

Therefore the result follows. □

Next, recall from Proposition 2.3.1(i) that for any $J \subseteq S$, we have (W_J, J) is a Coxeter system. Consequently, if W_J is finite, then the above discussion of largest elements apply.

Definition. Let (W,S) be a Coxeter system with finite parabolic subgroup W_J for some $J \subseteq S$. Then we denote $w_0(J)$ as the unique element in W_J of maximal length. We say $w_0(J)$ is the *largest element* in W_J.

Similarly, we can define the largest elements of W^J and ${}^JW$ analogously. This make sense because W^J and ${}^JW$ are directed posets under the Bruhat order (see Corollary 2.5.3, [BB05]).

Definition. Let (W,S) be a Coxeter system. If W^J is finite, then we denote w_0^J as the unique maximal element in W^J. Similarly, if ${}^JW$ is finite, then we denote ${}^Jw_0$ as the unique maximal element in ${}^JW$.

By attaching the necessary subscripts or superscripts, we get analogous results for all the above properties in terms of parabolic subgroups and quotients. In particular, we note that the length function $\ell(\cdot)$ must changed to $\ell_J(\cdot)$. Since we will deal with descent sets in later chapters, we emphasize the following analogous result:

Proposition 2.4.6. Let (W,S) be any Coxeter system, let $J \subseteq S$, and let $w \in W_J$. Then the following are equivalent:

(i) $D_L(w) = J$.

(ii) $D_R(w) = J$.

(iii) W_J is finite, and $w = w_0(J)$.

Proof: This is just Proposition 2.4.3 applied to Coxeter system (W_J, J). □

One very important relation between the various largest elements is the following:

$$w_0 = w_0^J \cdot w_0(J) = w_0(J) \cdot {}^Jw_0. \tag{2.15}$$

By considering lengths, we get the following useful identity:

$$\ell(w_0) = \ell(w_0^J) + \ell(w_0(J)) = \ell(w_0(J)) + \ell({}^Jw_0) \tag{2.16}$$

There is also a Chain Property analogous to Theorem 2.1.6:

Theorem 2.4.7. (**Chain Property**) If $u < w$ in W^J, then there exists a chain $u = u_0 \lhd u_1 \lhd \cdots \lhd u_k = w$. Similarly, if $u' < w'$ in ${}^JW$, then there exists a chain $u' = u'_0 \lhd u'_1 \lhd \cdots \lhd u'_{k'} = w'$.

Proof: See (Theorem 2.5.5, [BB05]). □

Corollary 2.4.8. All maximal chains in W^J and ${}^JW$ have the same length.

Proof: This is a direct consequence of the above Chain Property. □

Next, we apply the discussion of largest elements to descent classes. First we make the following observation:

Proposition 2.4.9. Let $I \subseteq J \subseteq S$, then the descent classes $\mathcal{D}_I^J$ and ${}_I^J\mathcal{D}$ are non-empty if and only if W_I is finite.

Proof: See (Theorem 6.2, [BW88]). □

Theorem 2.4.10. Let $I \subseteq J \subseteq S$, and let the parabolic subgroup W_I be finite. Then with respect to the Bruhat order, the following hold:

(i) $\mathcal{D}_I^J$ and ${}_I^J\mathcal{D}$ each has a least element $w_0(I)$.

(ii) $\mathcal{D}_I^J$ is finite if and only if $W^{S\backslash J}$ is finite. If so, then the largest element in $\mathcal{D}_I^J$ is $w_0^{S\backslash J}$.

(iii) ${}_I^J\mathcal{D}$ is finite if and only if ${}^{S\backslash J}W$ is finite. If so, then the largest element in ${}_I^J\mathcal{D}$ is ${}^{S\backslash J}w_0$.

Proof: See (Theorem 6.2, [BW88]). □

Corollary 2.4.11. Let $J \subseteq S$ such that W_J is finite. Then with respect to the Bruhat order, $w_0(J)$ is both the smallest word having left descent set J and the smallest word having right descent set J. If $W^{S\backslash J}$ is finite, then the largest word having right descent set J is $w_0^{S\backslash J}$. If ${}_I^J\mathcal{D}$ is finite, then the largest word having left descent set J is ${}^{S\backslash J}w_0$.

Proof: Substitute $i = j$ into Theorem 2.4.10 above. □

Chapter 3

Reduced Expressions and Braid Moves

One of the unifying themes in the study of combinatorial properties of Coxeter systems is the combinatorics of reduced expressions. Given a Coxeter system (W, S), each word $w \in W$ represents a class $\mathcal{R}(w)$ of reduced expressions, so in order to gain a better understanding of Coxeter systems and the combinatorics of descent sets, it is imperative that we 'get our hands dirty' and study how one reduced expression is obtained from another.

In this chapter, we shall introduce the Word Property and develop the theory of braid moves and sequences of braid moves. With the notable exception of the Word Property (Theorem 3.1.2), most of the other results in this chapter are new. As such, beyond the Word Property, we have to develop both the theory and the notations from scratch.

3.1 Word Property

Definition. Let (W, S) be a Coxeter system. If $|S| \geq 2$, and if $s, s' \in S$ are distinct generators, then we denote $\alpha_{s,s'}(k)$ as the expression $ss'ss'\cdots$ with expression length k. We call $\alpha_{s,s'}(k)$ an *alternating expression*.

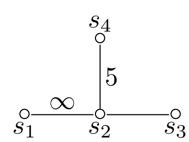
Recall from (1.3) that if $s, s' \in S$ are distinct and $m(s, s') \neq \infty$, then $\alpha_{s,s'}(m(s, s'))$ and $\alpha_{s',s}(m(s', s))$ represent the same word. Thus, for any given word $w \in W$ and any expression $s_1 \cdots s_k$ for w, if $\alpha_{s,s'}(m(s, s'))$ occurs as a sub-expression of $s_1 \cdots s_k$, then by replacing this sub-expression with $\alpha_{s',s}(m(s', s))$, the new expression obtained is still an expression for w. Since $m(s, s') = m(s', s)$, the expression length remains invariant under this replacement. In particular, if $s_1 \cdots s_k$ is a reduced expression, then the new expression obtained after the replacement must also be a reduced expression.

Also, we have $ss = e$ for all $s \in S$, so deleting any occurrence of ss from an expression does not change the word it represents, although the expression length of the new expression obtained is decreased by 2. In particular, if $s_1 \cdots s_k$ is a reduced expression, then for every $s \in S$, the expression $s_1 \cdots s_k$ must necessarily have no occurrence of ss as a sub-expression. These observations motivate us to define the following:

Definition. Let w_i denote the expression $s_1 \cdots s_k$ (not necessarily reduced). We define a *nil move* on w_i as the deletion of a sub-expression of the form ss from w_i (for some $s \in S$) . Also, we define a *braid move* on w_i as the replacement of a sub-expression of the form $\alpha_{s,s'}(m(s,s'))$ with the expression $\alpha_{s',s}(m(s',s))$ (for some distinct $s, s' \in S$ satisfying $m(s,s') \neq \infty$). If an expression w_i is changed to an expression w_i' by either a nil move or a braid move, then we write $w_i \sim w_i'$.

Example 3.1.1. Recall the Coxeter system (W,S) in Example 1.1.1. For the convenience of the reader, the corresponding Coxeter matrix and Coxeter diagram is reproduced here:

$$\begin{bmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 3 & 5 \\ 2 & 3 & 1 & 2 \\ 2 & 5 & 2 & 1 \end{bmatrix} \quad \longleftrightarrow$$



The following is then a valid sequence of two nil moves and two braid moves in (W,S):

$$s_2 s_3 \underline{s_1 s_1} s_2 s_4 s_3 \sim \underline{s_2 s_3 s_2} s_4 s_3 \sim s_3 s_2 \underline{s_3 s_4} s_3 \sim s_3 s_2 s_4 \underline{s_3 s_3} \sim s_3 s_2 s_4$$

Note that the sub-expressions involved in the nil moves and braid moves have been underlined for the convenience of the reader.

In view of the above discussion, nil moves and braid moves do not change the word that the expressions represent, so if v_0 is an expression for the word $v \in W$, and $v_0 \sim v_1 \sim \cdots \sim v_k$ is a sequence of nil moves and braid moves, then $v_0, v_1, \ldots, v_k$ are all expressions representing the same word v. In particular, if $v_0 \in \mathcal{R}(v)$, then all the moves are necessarily braid moves, and we get $v_0, v_1, \ldots, v_k \in \mathcal{R}(v)$.

We now come to the most important theorem in this chapter, for which many of the other results in the chapter are based upon:

Theorem 3.1.2. (**Word Property**) Let (W,S) be a Coxeter system, and let $w \in W$. Then the following hold:

(i) Any expression $s_1 \cdots s_k$ for w can be transformed into a reduced expression for w by a sequence of nil moves and braid moves.

(ii) Every two reduced expressions for w can be connected via a sequence of braid moves.

Proof: See (Theorem 3.3.1, [BB05]). □

Next, we shall introduce some useful notations:

Definition. Let (W, S) be a Coxeter system. Given any $v \in W$, let $v_i = s_{i_1} \cdots s_{i_k}$ be an expression (not necessarily reduced) for v. We say s_{i_j} is the j-th *coordinate* of the expression $v_i = s_{i_1} \cdots s_{i_k}$ for v, and we say the *index* of the specific letter s_{i_j} in v_i is j. Note that coordinates are letters among $\{s_{i_1}, \ldots, s_{i_k}\}$, while indices are integers in $[k]$. For any $t_1, t_2 \in [k]$, if $t_1 \le t_2$, then define $\tau(v_i, t_1 : t_2)$ to be the subexpression $s_{i_{t_1}} \cdots s_{i_{t_2}}$, and if $t_1 > t_2$, then set $\tau(v_i, t_1 : t_2)$ as the empty expression. If $t_1 = t_2 = t$, we simplify our notation and write $\tau(v_i, t)$ instead of $\tau(v_i, t : t)$ to denote the letter s_{i_t}. Also, define $\overline{\tau}(v_i, t_1 : t_2)$ to be the subexpression $s_{i_1} \cdots s_{i_{t_1 - 1}} s_{i_{t_2+1}} \cdots s_{i_k}$ obtained by deleting all the coordinates with indices in the range $[t_1, t_2]$, i.e. we get $\overline{\tau}(v_i, t_1 : t_2)$ after deleting the subexpression $\tau(v_i, t_1 : t_2)$ from v_i. Again, if $t_1 = t_2 = t$, we simplify our notation and write $\overline{\tau}(v_i, t)$ instead of $\overline{\tau}(v_i, t : t)$. If $t_1 > t_2$, we set $\overline{\tau}(v_i, t_1 : t_2)$ as the whole expression v_i.

Example 3.1.3. Denote v_i as the expression $s_1 s_3 s_2 s_5 s_2 s_4 s_1$. Then $\tau(v_i, 2 : 6) = s_3 s_2 s_5 s_2 s_4$, $\tau(v_i, 7) = s_1$, and $\tau(v_i, 3 : 1)$ is the empty expression, while we have $\overline{\tau}(v_i, 3 : 4) = s_1 s_3 s_2 s_4 s_1$, $\overline{\tau}(v_i, 2) = s_1 s_2 s_5 s_2 s_4 s_1$, and $\overline{\tau}(v_i, 6 : 3) = s_1 s_3 s_2 s_5 s_2 s_4 s_1$. The 2nd coordinate of v_i is s_3, the second occurrence of s_2 (from left to right) in v_i has index 5, and the index of (the only) s_5 in v_i is 4.

One simple but useful observation is the following:

Proposition 3.1.4. Any sub-expression of a reduced expression is reduced.

Proof: For any $v \in W$, let $v_i \in \mathcal{R}(v)$, and denote $\ell(v) = k$. Consider an arbitrary subexpression $w_i = \tau(v_i, t_1 : t_2)$, where$t_1, t_2 \in [k]$ satisfies $t_1 \le t_2$. Denote $u_i = \tau(v_i, 1 : t_1 - 1), u'_i = \tau(v_i, t_2 + 1, k)$, and let u, w, u' be the words representing the expressions u_i, w_i, u'_i respectively. Note that $\ell(u) \le t_1 - 1, \ell(w) \le t_2 - t_1 + 1, \ell(u') \le k - t_2$. By definition, $v_i = u_i w_i u'_i \in \mathcal{R}(v)$, hence $k = \ell(v_i) \le \ell(u) + \ell(w) + \ell(u')$ by Proposition 1.1.4. Suppose $w_i \notin \mathcal{R}(w)$, then $\ell(w) < t_2 - t_1 + 1$ implies $\ell(u) + \ell(w) + \ell(u') < (t_1 - 1) + (t_2 - t_1 + 1) + (k - t_2) = k$, which is a contradiction. Consequently, w_i must be a reduced expression. $\square$

3.2 Sequences of Braid Moves

In this section, we focus our attention on braid moves and sequences of braid moves. For any given word $v \in W$, and let $v_1 = s_{i_1} \cdots s_{i_m}$ be an expression (not necessarily reduced) for v. Assume $t = m(s, s')$ is finite for some distinct $s, s' \in S$ (we necessarily have $t \ge 2$) and suppose the subexpresion $\tau(v_1, k : k + t - 1)$ is the alternating expression $\alpha_{s,s'}(t)$ for some $k \in [m - t + 1]$. Denote $s_{i'_k} \cdots s_{i'_{k+t-1}}$ as the alternating expression $\alpha_{s',s}(t)$. We then get

$$v_2 = s_{i_1} \cdots s_{i_{k-1}} s_{i'_k} \cdots s_{i'_{k+t-1}} s_{i_{k+t}} \cdots s_{i_m}$$

is another expression for v, which can be obtained by a braid move $v_1 \sim v_2$.

Definition. We shall denote the above braid move just discussed by

$$s_{i_1} \cdots s_{i_m} \xrightarrow{[k,k+t-1]} s_{i_1} \cdots s_{i_{k-1}} s_{i'_k} \cdots s_{i'_{k+t-1}} s_{i_{k+t}} \cdots s_{i_m} \ , \tag{3.1}$$

or more briefly, $v_1 \xrightarrow{[k,k+t-1]} v_2$, where k and $k+t-1$ are the first and last indices respectively of the subexpression to be replaced for the braid move.

Remark. The notation $[k, k+t-1]$ is suggestive of the closed interval $[k, k+t-1] \subseteq \mathbb{R}$, and for any positive integers $k_1 < k_2$, we shall write without ambiguity $t \in [k_1, k_2]$ to refer to t being contained in the closed interval $[k_1, k_2]$, as well as t being one of the indices involved in the braid move represented by $[k_1, k_2]$ ($t \notin [k_1, k_2]$ is analogously defined). Each braid move can then be identified by some closed interval $[k_1, k_2]$.

Example 3.2.1. If $a, b, c \in S$ are distinct, such that $m(a, b) = 3, m(b, c) = 5, m(a, c) = 2$, then the following is a valid sequence of braid moves:

$$cb\underline{aba}cbcba \xrightarrow{[3,5]} cbba\underline{bcbcb}a \xrightarrow{[5,9]} cbbacbcb\underline{ca} \xrightarrow{[9,10]} cbbacbcbac.$$

One useful observation is the following:

Proposition 3.2.2. Any alternating expression $\alpha_{s,s'}(t)$ that is a sub-expression of a reduced expression must satisfy $t \leq m(s, s')$.

Proof: For an arbitrary reduced expression $v_i = s_{i_1} \cdots s_{i_k}$, assume $\tau(v_i, t_1 : t_2) = \alpha_{s,s'}(t)$ for some $t_1, t_2 \in [k]$ satisfying $t_2 - t_1 + 1 = t$. Denote $T = m(s, s')$ and suppose on the contrary that $t > T$. Denote $s'_{i_{t_1}} \cdots s'_{i_{t_1+T-1}}$ as the expression $\alpha_{s',s}(T)$. We can then apply the braid move $v_i \xrightarrow{[t_1:t_1+T-1]} v'_i$, where v'_i is the expression $s_{i_1} \cdots s_{i_{t_1-1}} s'_{i_{t_1}} \cdots s'_{i_{t_1+T-1}} s_{i_{t_1+T}} \cdots s_{i_k}$. We then get $s'_{i_{t_1+T-1}} = s_{i_{t_1+T}}$, so v'_i is not reduced, which contradicts the fact that braid moves applied to reduced expressions yield reduced expressions. □

Definition. For any $v \in W$, denote $\Phi(v)$ as the collection of all (finite) sequences of braid moves of reduced expressions for v. Note that any $\phi \in \Phi(v)$ can be represented by

$$\phi : v_0 \xrightarrow{[a_1,b_1]} v_1 \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} v_N$$

for some $v_0, v_1, \ldots, v_N \in \mathcal{R}(v)$, where for each $t \in [N]$, a_t and b_t are integers satisfying $1 \leq a_t < b_t \leq \ell(v)$. We can then identify each $\phi \in \Phi(v)$ by $(v_0, v_N, \vec{a}, \vec{b})$, where $\vec{a} = (a_1, \ldots, a_N), \vec{b} = (b_1, \ldots, b_N)$ are N-tuples in $\mathbb{Z}^N$.

Corollary 3.2.3. Let $v \in W$, and let $\phi \in \Phi(v)$. Then any two consecutive braid moves in ϕ are either disjoint, equal, or intersect in exactly one coordinate. More explicitly, if $v_0, v_1, v_2 \in \mathcal{R}(v)$ such that $v_0 \xrightarrow{[a_1,b_1]} v_1 \xrightarrow{[a_2,b_2]} v_2$ is a valid sequence of braid moves, then exactly one of the following is true:

(i) $[a_1, b_1] \cap [a_2, b_2] = \emptyset$.

(ii) $[a_1, b_1] = [a_2, b_2]$.

(iii) $b_1 = a_2$.

(iv) $a_1 = b_2$.

Proof: Since a braid move involves the replacement of an alternating expression, it follows from the definition of an alternating sequence that $a_1 < b_1$ and $a_2 < b_2$, hence the above four possibilities are mutually disjoint. If $[a_1, b_1] \cap [a_2, b_2] = \emptyset$, then we are done. If $|[a_1, b_1] \cap [a_2, b_2]| = 1$, then either (iii) or (iv) is true, and we are done. If $|[a_1, b_1] \cap [a_2, b_2]| \geq 2$, then there must be two consecutive integers $m, m+1$ contained in $[a_1, b_1] \cap [a_2, b_2]$. By denoting $\tau(v_0, m) = s, \tau(v_0, m+1) = s'$, we must then have $s, s' \in S$ are distinct generators satisfying $m(s, s') \neq \infty$, and each of $\tau(v_0, a_1 : b_1)$ and $\tau(v_0, a_2 : b_2)$ must be one of the alternating expressions $\alpha_{s,s'}(m(s, s'))$ or $\alpha_{s',s}(m(s, s'))$. Consequently, it easily follows from Proposition 3.2.2 that we must then have $[a_1, b_1] = [a_2, b_2]$. $\square$

Definition. Given $v \in W$, let $v_i, v_i' \in \mathcal{R}(v)$. For any $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, we define the *sequence length* of ϕ as the length of $\vec{a}$, or equivalently, the length of $\vec{b}$, and we denote this sequence length as $\widetilde{\ell}(\phi)$. More explicitly, if we denote v_i, v_i' as v_{i_0} and v_{i_N} respectively, and if ϕ is given by

$$\phi : v_{i_0} \xrightarrow{[a_1,b_1]} v_{i_1} \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} v_{i_N},$$

then the sequence length of ϕ is N. We define $\ell(v_i, v_i')$ as the minimal sequence length of all the possible sequences of braid moves from v_i to v_i'. This is well-defined by the Word Property (Theorem 3.1.2). If $\widetilde{\ell}(\phi) = \ell(v_i, v_i')$, then we say ϕ is *reduced*, or ϕ is a *reduced sequence of braid moves*. We shall also denote $\ell(\phi)$ to mean $\widetilde{\ell}(v_i, v_i')$, so that $\ell(\phi) = \widetilde{\ell}(\phi)$ if and only if ϕ is reduced. Also, we say $\ell(\phi)$ is the *length* of the sequence ϕ.

Remark. Observe that for any $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, if we are given $v_i, \vec{a}$ and $\vec{b}$, then we can uniquely determine v_i', and if we are given $v_i', \vec{a}, \vec{b}$, then we can uniquely determine v_i. Also, the sequence length $\ell(\phi)$ can be determined by the length of the tuples $\vec{a}, \vec{b}$. This notation for a sequence of braid moves would be useful if we need to refer to some sequence of braid moves from v_0 to v_N without needing to specify the intermediate reduced expressions in the sequence.

Definition. Let $\phi_1 = (u_i, u_i', \vec{a}, \vec{b}), \phi_2 = (v_i, v_i', \vec{a}', \vec{b}')$ be elements of $\Phi(v)$. If $u_i = u_i'$ and $v_i = v_i'$ and $\ell(\phi_1) = \ell(\phi_2)$, then we write $\phi_1 \equiv \phi_2$, and we say that the two sequences of braid moves are *equivalent*. It is easy to check that $\equiv$ defines an equivalence relation.

3.3 Boundary Pairs

In this section, we shall introduce the notion of boundary pairs. We shall see that for any word $w \in W$, even if $\mathcal{R}(w)$ is very large, there are certain restrictions on the reduced expressions given by boundary pairs, and we shall study the consequences.

Lemma 3.3.1. Given a word $v \in W$, let $\ell(v) = m$, and let $v_0, v_1, \ldots, v_N \in \mathcal{R}(v)$ such that we have the following sequence of braid moves:

$$v_0 \xrightarrow{[a_1,b_1]} v_1 \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} v_N\ .$$

If $k \in [m]$ such that $k \notin [a_t, b_t]$ for all $t \in [N]$, then for all $i, j \in \{0, 1, \ldots, N\}$, we have the following:

(i) $\tau(v_i, k) = \tau(v_j, k)$.

(ii) $\tau(v_i, 1 : k-1)$ and $\tau(v_j, 1 : k-1)$ represent the same word (in the case $k > 1$).

(iii) $\tau(v_i, k+1 : m)$ and $\tau(v_j, k+1 : m)$ represent the same word (in the case $k < m$).

Equivalently, the k-th coordinate of the expression, the word representing the subexpression formed by the first $(k-1)$ coordinates (if any), and the word representing the subexpression formed by the last $(m-k)$ coordinates (if any), remain invariant in this sequence of braid moves.

Proof: Since $k \notin [a_t, b_t]$ for all $t \in [N]$, we either have $a_t < b_t \leq k-1$ or $k+1 \leq a_t < b_t$ for each $t \in [N]$. The k-th coordinate of each expression $v_0, v_1, \ldots, v_N$ is not involved in any braid move and hence must remain invariant. By deleting the last $(k-m+1)$ coordinates of each expression and omitting any braid moves $[a_t, b_t]$ such that $k+1 \leq a_t < b_t$, we get a sequence of braid moves for the subexpressions formed by the first $(k-1)$ coordinates of each $v_0, v_1, \ldots, v_N$. Similarly, by deleting the first k coordinates of each expression and omitting any braid moves $[a_t, b_t]$ such that $a_t < b_t \leq k-1$, we get another sequence of braid moves for the subexpressions formed by the last $(m-k)$ coordinates of each $v_0, v_1, \ldots, v_N$. Since braid moves do not change the word, the result follows. $\square$

Definition. Let $v \in W$, let $v_i, v_i' \in \mathcal{R}(v)$, and let $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, where $\vec{a} = (a_1, \ldots, a_N), \vec{b} = (b_1, \ldots, b_N)$ are N-tuples in $\mathbb{Z}^N$. If $k \in [\ell(v)-1]$ such that for all $t \in [N]$, we have $\{k, k+1\} \not\subseteq [a_t, b_t]$, then we say $\{k, k+1\}$ is a *boundary pair* of ϕ, k is a *right boundary coordinate* of ϕ, and $k+1$ is a *left boundary coordinate* of ϕ. The reasons for these notations used will be apparent later. We define $K(\phi), K_R(\phi)$ and $K_L(\phi)$ to be the sets of boundary pairs of ϕ, right boundary coordinates of ϕ, and left boundary coordinates of ϕ respectively. More explicitly, we have the following:

$$\begin{aligned} K(\phi) &= \{\{k, k+1\} \mid k \in [\ell(v)-1], \{k, k+1\} \not\subseteq [a_t, b_t] \ \forall t \in [N]\}, && (3.2) \\ K_R(\phi) &= \{k \in [\ell(v)-1] \mid \{k, k+1\} \not\subseteq [a_t, b_t] \ \forall t \in [N]\}, && (3.3) \\ K_L(\phi) &= \{k+1 \in [\ell(v)-1] \mid \{k, k+1\} \not\subseteq [a_t, b_t] \ \forall t \in [N]\}. && (3.4) \end{aligned}$$

Theorem 3.3.2. Given a word $v \in W$, let $\ell(v) = m \geq 2$ and let $v_0, v_1, \ldots, v_N \in \mathcal{R}(v)$ such that we have the following sequence of braid moves:

$$\phi : v_0 \xrightarrow{[a_1,b_1]} v_1 \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} v_N \ .$$

If $\{k, k+1\}$ is a boundary pair of ϕ, then for all $i, j \in \{0, 1, \ldots, N\}$, we have the following:

(i) $\tau(v_i, 1 : k)$ and $\tau(v_j, 1 : k)$ represent the same word.

(ii) $\tau(v_i, k+1 : m)$ and $\tau(v_j, k+1 : m)$ represent the same word.

Proof: Since $k, k+1$ cannot be both contained in any of $[a_1, b_1], \ldots, [a_N, b_N]$, we either have $a_t < b_t \leq k$ or $k+1 \leq a_t < b_t$ for each $t \in [N]$. This means $k \in [a_t, b_t]$ if and only if $b_t = k$ and $k+1 \in [a_t, b_t]$ if and only if $a_t = k+1$. Let $t_1 < \ldots < t_r$ be all the distinct integers in $[m]$ (if any) such that $a_{t_i} = k+1$ for each $i \in [r]$. For each $i \in [r]$ and each braid move $v_{t_i-1} \xrightarrow{[a_{t_i}, b_{t_i}]} v_{t_i}$, since $k \notin [a_{t_i}, b_{t_i}]$, Lemma 3.3.1 gives us $\tau(v_{t_i-1}, 1:k) = \tau(v_{t_i}, 1:k)$ and $\tau(v_{t_i-1}, k+1:m) = \tau(v_{t_i}, k+1:m)$. Next, consider the following $r+1$ (possibly empty) sequences of braid moves:

$$\begin{array}{ll} v_0 \xrightarrow{[a_1,b_1]} \cdots \xrightarrow{[a_{t_1-1},b_{t_1-1}]} v_{t_1-1}\ , & \\ v_{t_i} \xrightarrow{[a_{t_i+1},b_{t_i+1}]} \cdots \xrightarrow{[a_{t_{i+1}-1},b_{t_{i+1}-1}]} v_{t_{i+1}-1}\ , & \text{for } i \in [r-1] \\ v_r \xrightarrow{[a_{r+1},b_{r+1}]} \cdots \xrightarrow{[a_N,b_N]} v_N\ . & \end{array}$$

For each such sequence, $k+1$ is not involved in any of the braid moves, hence by Lemma 3.3.1, the word representing the subexpression formed by the first k coordinates and the word representing the subexpression formed by the last $(m-k)$ coordinates both remain invariant in each of these sequences of braid moves. The result follows. $\square$

Remark. The above theorem tells us that given any $v \in W$ and any $\phi = (v_i, v_i', \vec{a}, \vec{v}) \in \Phi(v)$, if k is a right boundary coordinate of ϕ, then the word representing the subexpression formed by the first k coordinates remains invariant in the sequence ϕ of braid moves, so if we delete all coordinates to the right of the k-th coordinate and omit all braid moves $[a_t, b_t]$ such that $a_t > k$, then we get another valid sequence of braid moves from the word formed by the first k coordinates of v_i to the word formed by the first k coordinates of v_i'. Similarly, if $k+1$ is a left boundary coordinate of ϕ, then the word representing the subexpression formed by the last $(m-k)$ coordinates remains invariant in ϕ, so if we delete all coordinates to the left of the $(k+1)$-th coordinate and omit all braid moves $[a_t, b_t]$ such that $b_t < k+1$, then we get another valid sequence of braid moves from the word formed by the last $\ell(v)-k$ coordinates of v_i to the word formed by the last $\ell(v)-k$ coordinates of v_i'. We shall first make another definition, then record this observation as Corollary 3.3.3.

Definition. Let $v \in W$ and let $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, where $\vec{a} = (a_1, \ldots, a_N), \vec{b} = (b_1, \ldots, b_N)$. Suppose $\{k, k+1\}$ is a boundary pair of ϕ. Let $\{i_1, \ldots, i_r\}$ be the set of all integers in $[N]$ satisfying $i_1 < \ldots < i_r$ and satisfying $b_{i_t} \leq k$ for each $t \in [r]$. Write $[N] \setminus \{i_1, \ldots, i_r\}$ as $\{j_1, \ldots, j_{N-r}\}$ so that $j_1 < \ldots < j_{N-r}$. By the definition of a boundary pair, we know that $\{j_1, \ldots, j_{N-r}\}$ is the set of all integers in $[N]$ such that $a_{j_t} \geq k+1$ for each $t \in [N-r]$. We shall then define the following tuples:

$$\begin{array}{rclrcl} \vec{a}_{\leq k} & = & (a_{i_1}, \ldots, a_{i_r}), & \vec{b}_{\leq k} & = & (b_{i_1}, \ldots, b_{i_r}), \\ \vec{a}_{\geq k+1} & = & (a_{j_1}, \ldots, a_{j_{N-r}}), & \vec{b}_{\geq k+1} & = & (b_{j_1}, \ldots, b_{j_{N-r}}). \end{array}$$

In other words, $\vec{a}_{\leq k}$ and $\vec{b}_{\leq k}$ are obtained from $\vec{a}$ and $\vec{b}$ respectively by deleting all a_t, b_t that are not $\leq k$, while $\vec{a}_{\geq k+1}$ and $\vec{b}_{\geq k+1}$ are obtained from $\vec{a}$ and $\vec{b}$ respectively by deleting all a_t, b_t that are not $\geq k+1$.

Corollary 3.3.3. For any $v \in W$, let $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, and suppose $\{k, k+1\}$ is a boundary pair of ϕ. Denote $\vec{a} = (a_, \ldots, a_N)$ and $\vec{b} = (b_1, \ldots, b_N)$. Denote

$u_i = \tau(v_i, 1:k), w_i = \tau(v_i, k+1:\ell(v)), u_i' = \tau(v_i', 1:k), w_i' = \tau(v_i', k+1:\ell(v))$ so that $v_i = u_i w_i$ and $v_i' = u_i' w_i'$. Let u be the word represented by the expression u_i and let w be the word represented by the expression w_i. Then we have the following:

(i) $(u_i, u_i', \vec{a}_{\leq k}, \vec{b}_{\leq k}) \in \Phi(u)$.

(ii) $(w_i, w_i', \vec{a}_{\geq k+1}, \vec{b}_{\geq k+1}) \in \Phi(w)$.

Proof: Most of this has already been proven in the discussion earlier. We only need to check that u_i, u_i' are reduced expressions for u and w_i, w_i' are reduced expressions for w. This is true, since any sub-expression of a reduced expression is reduced by Proposition 3.1.4. □

Next, we shall investigate what happens when we swap a pair of braid moves in a sequence of braid moves.

Lemma 3.3.4. Let $v \in W$, let $v_1, v_2, v_3 \in \mathcal{R}(v)$, and let $\phi : v_1 \xrightarrow{[a_1,b_1]} v_2 \xrightarrow{[a_2,b_2]} v_3$ be a sequence of braid moves. If $b_1 < a_2$ or $b_2 < a_1$, then $\phi' : v_1 \xrightarrow{[a_2,b_2]} v_2' \xrightarrow{[a_1,b_1]} v_3$ is also a valid sequence of braid moves for some other $v_2' \in \mathcal{R}(v)$. In particular, we have $\phi \equiv \phi'$.

Proof:
If $b_1 < a_2$, then $\{b_1, b_1+1\}$ is a boundary pair. Denote $u_i = \tau(v_i, 1:b_1), w_i = \tau(v_i, b_1+1:\ell(v)), u_i' = \tau(v_i', 1:b_1), w_i' = \tau(v_i', b_1+1:\ell(v))$ so that $v_i = u_i w_i$ and $v_i' = u_i' w_i'$. Let u be the word represented by the expression u_i and let w be the word represented by the expression w_i. By Corollary 3.3.3, we get two sequences of braid moves $u_i \xrightarrow{[a_1,b_1]} u_i'$ and $w_i \xrightarrow{[a_2,b_2]} w_i'$, hence by adjoining the expressions, we get the sequence of braid moves $u_i w_i \xrightarrow{[a_2,b_2]} u_i w_i' \xrightarrow{[a_1,b_1]} u_i' w_i'$. Note that $u_i w_i, u_i w_i', u_i' w_i' \in \mathcal{R}(uw) = \mathcal{R}(v)$, hence we are done with this case. The other case $b_2 < a_1$ can also be proven using a similar argument. □

Consequently, for any $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, where $\vec{a} = (a_1, \ldots, a_N), \vec{b} = (b_1, \ldots, b_N)$ are N-tuples in $\mathbb{Z}^N$, if $\{k, k+1\}$ is a boundary pair of ϕ, then by repeated uses of Corollary 3.3.3, we can rearrange the braid moves so that we get a new sequence of braid moves $\phi' = (v_i, v_i', \vec{a}', \vec{b}')$, such that the following conditions hold:

(i) $\phi' \equiv \phi$.

(ii) $\vec{a}' = (a_1', \ldots, a_N')$ is a rearrangement of $\vec{a} = (a_1, \ldots, a_N)$.

(iii) $\vec{b}' = (b_1', \ldots, b_N')$ is a rearrangement of $\vec{b} = (b_1, \ldots, b_N)$.

(iv) There is some $N_0 \in \{0, 1, \ldots, N\}$ such that $b_1', \ldots, b_{N_0}' \leq k$ and $a_{N_0+1}', \ldots, a_N' \geq k+1$.

In other words, we can always rearrange the braid moves in ϕ to get ϕ', such that the first N_0 braid moves in ϕ' only involve the first k coordinates, and the last $N - N_0$ braid moves in ϕ' only involve the last $\ell(v) - k$ coordinates.

By considering all possible boundary pairs of ϕ, we can then rearrange the braid moves in ϕ to get a new sequence $\phi'' = (v_i, v_i', \vec{a}'', \vec{b}'') \in \Phi(v)$, such that for every left boundary coordinate k of ϕ, there exists some $N_k \in \{0, 1, \ldots, N\}$ such that the first N_k braid moves in ϕ'' only involve the first k coordinates, and the last $N - N_k$ braid moves in ϕ'' only involve the last $\ell(v) - k$ coordinates. More explicitly, we must have $\vec{a}'' = (a_1'', \ldots, a_N''), \vec{b}'' = (b_1'', \ldots, b_N'')$ satisfy any of the following equivalent conditions:

(i) If k is a right boundary coordinate of ϕ and $b_t'' \leq k < a_{t'}''$ for some $t, t' \in [N]$, then $t < t'$.

(ii) If k is a right boundary coordinate of ϕ and $a_t'' < k < a_{t'}''$ for some $t, t' \in [N]$, then $t < t'$.

(iii) If k is a left boundary coordinate of ϕ and $b_t'' < k \leq a_{t'}''$ for some $t, t' \in [N]$, then $t < t'$.

(iv) If k is a left boundary coordinate of ϕ and $b_t'' < k < b_{t'}''$ for some $t, t' \in [N]$, then $t < t'$.

The equivalence of the above four conditions follows from the definition of left and right boundary coordinates.

Definition. Following the notations above, we say that ϕ is *normalized* to ϕ'', and we say the sequence ϕ'' is the *normalized sequence of* ϕ. Any arbitrary sequence is called *normal* if it equals its normalized sequence.

Next, we note that for any $v \in W$ and any $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, if we order the elements in $K_R(\phi) = \{k_1, \ldots, k_r\}$ so that $k_1 < \ldots < k_r$, then every braid move $[a_t, b_t]$ of ϕ must satisfy $k_i < a_t < b_t \leq k_{i+1}$ for some $i \in [r-1]$, i.e. $[a_t, b_t] \subseteq (k_i, k_{i+1}]$. This gives us a natural way to group the braid moves, which motivates the following definition:

Definition. Let $v \in W$, let $\phi = (v_i, v_i', \vec{a}, \vec{b}) \in \Phi(v)$, where $\vec{a} = (a_1, \ldots, a_N), \vec{b} = (b_1, \ldots, b_N)$, and let $K_R(\phi) = \{k_1, \ldots, k_r\}$ such that $k_1 < \ldots < k_r$. For each $i \in [r-1]$, if the set of closed intervals

$$A_i = \{[a_t, b_t] : t \in [N], [a_t, b_t] \subseteq (k_i, k_{i+1}]\}$$

is non-empty, then we say that A_i is a *braid move component* of ϕ. If ϕ has only one braid move component, then we say ϕ is *connected*.

Chapter 4

Comparisions of Descent Sets

In this chapter, we shall apply the theory developed in the previous three chapters to obtain results regarding descent sets. In Section 4.1, we introduce the idea of tagging a word and investigate what attaching two words together does to the corresponding descent set. In Section 4.2, we shall define the notion of dominating sets and derive results regarding dominating sets. All the results in this chapter are new, and we shall discuss the applications of these new results in Chapter 5.

4.1 Attaching and Tagging Elements

Suppose we know the elements in the descent sets $D_R(v)$ and $D_R(w)$ for some $v, w \in W$. What can we say about the elements in $D_R(vw)$? In this section, we shall introduce the idea of tagging a word, and then apply it to solve this question.

To motivate the idea of tagging a word, we first give an informal discussion, before we formalize the idea rigorously:

Recall that a braid move is by definition the replacement of an alternating expression $\alpha_{s,s'}(m(s,s'))$ with the alternating expression $\alpha_{s',s}(m(s',s))$, with the assumption that $s, s' \in S$ are distinct generators satisfying $m(s,s') \neq \infty$. Given a word $w \in W$, suppose $w_i \in \mathcal{R}(w)$ contains a sub-expression $\alpha_{s,s'}$, where $s, s' \in S$ are distinct generators satisfying $m(s,s') \neq \infty$. Consider this alternating sub-expression $\alpha_{s,s'}$. Circle the left-most coordinate of $\alpha_{s,s'}$ and box up the sub-expression formed by the remaining coordinates in $\alpha_{s,s'}$. We can then interpret a braid move as the swapping of the circle and the box, keeping the sub-expression in the box invariant, and letting the letter in the circle be t, where

$$t = \begin{cases} s, & \text{if } m(s,s') \equiv 0 \pmod 2 \\ s', & \text{if } m(s,s') \equiv 1 \pmod 2 \end{cases}. \tag{4.1}$$

Diagramatically, we have the following:

$$s\ \underbrace{\boxed{s'ss's\cdots}}_{\substack{\text{expression length}\\ m(s,s')-1}} \xrightarrow{\text{braid move}} \underbrace{\boxed{s'ss's\cdots}}_{\substack{\text{expression length}\\ m(s',s)-1}}\ t$$

By symmetry, we could instead have circled the right-most coordinate of $\alpha_{s,s'}$ and boxed up the sub-expression formed by the remaining coordinates in $\alpha_{s,s'}$. The coordinate in the circle would then be t (as defined in (4.1) above), so a braid move could also be represented diagramatically by the following:

$$\underbrace{\boxed{s'ss's\cdots}}_{\substack{\text{expression length}\\ m(s,s')-1}}\ t \xrightarrow{\text{braid move}} s\ \underbrace{\boxed{s'ss's\cdots}}_{\substack{\text{expression length}\\ m(s',s)-1}}$$

Suppose we "tag" the circle and keep track of where the circle moves to after braid moves are applied. Any braid move not involving this circle would not change the position of the circle relative to the expression for the given word w, while any braid move involving this circle would swap the circle with a sub-expression of expression length $m(s,s')-1$ for some distinct $s,s' \in S$ satisfying $m(s,s') \neq \infty$.

Next, consider a sequence of braid moves of sequence length N:

$$w_i = w_{i_0} \xrightarrow{[a_1,b_1]} w_{i_1} \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} w_{i_N}$$

Starting with $w_i = w_{i_0}$, we keep track of the relative position of the tagged circle relative to the various reduced expressions of w_{i_m} for w. Denote v_i as the expression formed by deleting the circled coordinate from w_i, and denote v as the word that v_i represents. We then observe that if we delete the coordinate in the tagged circle for each of the reduced expressions $w_{i_0}, w_{i_1}, \ldots, w_{i_N}$ in the above sequence, each of the resultant expressions is actually a reduced expression for v. This means that if we delete the circled coordinates from the reduced expressions $w_{i_0}, w_{i_1}, \ldots, w_{i_N}$, omit all braid moves involving the circled coordinates, and make necessary adjustments to the numbering of the indices, we would get a sequence of braid moves for the word v.

Keeping in mind the above informal discussion, we shall now rigorize the idea:

Suppose we are given $v, w \in W$ such that $v \lhd w$. By definition, we can write $v = wt$ for some $t \in T_R(w)$, so that $\ell(wt) + 1 = \ell(w)$. By Corollary 1.2.5, any reduced expression w_i for w has a unique coordinate whose deletion yields an expression v_i for v. By length considerations, we necessarily have $v_i \in \mathcal{R}(v)$. More explicitly, if $s_1 \cdots s_k \in \mathcal{R}(w)$, then there exists a unique $j \in [k]$ such that $\overline{\tau}(w_i, j) \in \mathcal{R}(v)$. Consequently, the following definition is well-defined:

Definition. Let $v, w \in W$ such that $v \lhd w$. For each $w_i \in \mathcal{R}(w)$, let $j \in [\ell(w)]$ be the unique integer such that $\overline{\tau}(w_i, j) \in \mathcal{R}(v)$. We then say the letter $\tau(w_i, j)$ is the *tagged letter of* w_i *with respect to covering* $v \lhd w$. Also, we say the index j is the *tag of* w_i *with respect to covering* $v \lhd w$, and we denote this as $\P_{v \lhd w}(w_i) = j$. If the context of the Bruhat covering $v \lhd w$ is clear, we simply say $\tau(w_i, j)$ is the *tagged letter*, and we say j is the *tag*.

Proposition 4.1.1. Let $v, w \in W$ such that $v \lhd w$. Let $w_i \in \mathcal{R}(w)$, and let $\P_{v\lhd w}(w_i) = j$. Then for any valid braid move $w_i \xrightarrow{[a_1,b_1]} w_i'$ satisfying $j \in [a_1, b_1]$, we either have $j = a_1$ or $j = b_1$.

Proof: Suppose not, then $b_1 - a_1 \geq 2$ and $a_1 + 1 \leq j \leq b_1 - 1$. Denote $\tau(w_i, j) = s$, denote $\tau(w_i, j-1) = s'$, and note that s, s' must be distinct, since w_i is reduced by assumption. By the definition of a braid move, the sub-expression $\tau(w_i, a_1 : b_1)$ is an alternating expression, so since this sub-expression contains the letters s and s', we must have $m(s, s') = m(s', s) = b_1 - a_1 + 1$, and we must either have $\tau(w_i, a_1 : b_1) = \alpha_{s,s'}(b_1 - a_1 + 1)$ or $\tau(w_i, a_1 : b_1) = \alpha_{s',s}(b_1 - a_1 + 1)$. Either case, since $j+1 \in [a_1, b_1]$, we get $\tau(w_i, j+1) = s'$.

Denote the expression $\overline{\tau}(w_i, j)$ as v_i. By the definition of a tag, v_i is a reduced expression for v. However, we observe that $\tau(v_i, j-1 : j) = s's'$, which implies v_i cannot be reduced, hence a contradiction. The result then follows. □

Proposition 4.1.2. Let $v, w \in W$ such that $v \lhd w$. Let $w_i \in \mathcal{R}(w)$, let $\P_{v\lhd w}(w_i) = j$, and let $\tau(w_i, j) = s$. Let $w_i \xrightarrow{[a_1,b_1]} w_i'$ be a valid braid move. If $j = a_1$, then by denoting $s' = \tau(w_i, j+1)$, we have $m(s, s') = b_1 - a_1 + 1$, and we have $\P_{v\lhd w}(w_i') = b_1$, with corresponding tagged letter

$$t = \begin{cases} s, & \text{if } b_1 - a_1 + 1 \equiv 0 \pmod 2 \\ s', & \text{if } b_1 - a_1 + 1 \equiv 1 \pmod 2 \end{cases}.$$

Similarly, if $j = b_1$, then by denoting $s'' = \tau(w_i, j-1)$, we have $m(s, s'') = b_1 - a_1 + 1$, and we have $\P_{v\lhd w}(w_i') = a_1$, with corresponding tagged letter

$$t = \begin{cases} s, & \text{if } b_1 - a_1 + 1 \equiv 0 \pmod 2 \\ s'', & \text{if } b_1 - a_1 + 1 \equiv 1 \pmod 2 \end{cases}.$$

Proof: Suppose $j = a_1$. By the definition of a braid move, we know $\tau(w_i, a_1 : b_1)$ is an alternating expression. Since $\tau(w_i, a_1) = \tau(w_i, j) = s$, and since $\tau(w_i, j+1) = \tau(w_i, a_1 + 1) = s'$, we necessarily have that $\tau(w_i, a_1 : b_1) = \alpha_{s,s'}(m(s, s'))$, so by comparing expression lengths, we get $m(s, s') = b_1 - a_1 + 1$. Denote $v_i = \overline{\tau}(w_i, a_1)$. By the definition of a tag, we have $v_i \in \mathcal{R}(v)$. We check that $\overline{\tau}(w_i', b_1)$ is the same expression as v_i, thus we get $\overline{\tau}(w_i', b_1) \in \mathcal{R}(v)$. Consequently, by the uniqueness of a tag, we get $\P_{v\lhd w}(w_i') = b_1$. Finally, by the definition of a braid move, we have $\tau(w_i', a_1 : b_1) = \alpha_{s',s}$, thus we get

$$\tau(w_i', b_1) = \begin{cases} s, & \text{if } b_1 - a_1 + 1 \equiv 0 \pmod 2 \\ s', & \text{if } b_1 - a_1 + 1 \equiv 1 \pmod 2 \end{cases}.$$

This proves the assertion for the case $j = a_1$. The case $j = b_1$ can be proven by a very similar argument. □

Proposition 4.1.3. Let $v, w \in W$ such that $v \lhd w$. Let $w_i \in \mathcal{R}(w)$, and let $\P_{v\lhd w}(w_i) = j$. Then for any valid braid move $w_i \xrightarrow{[a_1,b_1]} w_i'$ such that $j \notin [a_1, b_1]$, we have $\P_{v\lhd w}(w_i') = j$, and $\overline{\tau}(w_i', j) = \overline{\tau}(w_i, j) \in \mathcal{R}(v)$.

Proof: Denote $\ell(w) = k$. By Lemma 3.3.1, we get $\tau(w_i, 1 : j-1)$ and $\tau(w'_i, 1 : j-1)$ represent the same word, which we shall denote as u_i. Also, we get $\tau(w_i, j+1 : k)$ and $\tau(w'_i, j+1 : k)$ represent the same word, which we shall denote as v_i. It then follows that $\overline{\tau}(w_i, j) = u_i v_i = \overline{\tau}(w'_i, j)$. By the definition of a tag, we have $\overline{\tau}(w_i, j) \in \mathcal{R}(v)$, which implies $\overline{\tau}(w'_i, j) \in \mathcal{R}(v)$. Consequently, it follows from the uniqueness of the tag that $\P_{v \lhd w}(w'_i) = j$, and we are done. $\square$

Corollary 4.1.4. Let $v, w \in W$ such that $v \lhd w$. Let

$$w_{i_0} \xrightarrow{[a_1,b_1]} w_{i_1} \xrightarrow{[a_2,b_2]} \cdots \xrightarrow{[a_N,b_N]} w_{i_N}$$

be a sequence of braid moves such that any of (and hence all of) $w_{i_0}, \ldots, w_{i_N}$ are reduced expressions for w. For each $k \in \{0, 1, \ldots, N\}$, denote $\P_{v \lhd w}(w_{i_k}) = j_k$, and denote v_k as the expression $\overline{\tau}(w_{i_k}, j_k)$. Let $t_0, t_2, \ldots, t_m$, with $t_0 < t_1 < \ldots < t_m$, be all the distinct integers in $\{0, 1, \ldots, N-1\}$ (if any) such that $j_{t_r} \notin [a_{t_r+1}, b_{t_r+1}]$ for every $r \in \{0, 1, \ldots, m\}$. For each $r \in \{0, 1, \ldots, m-1\}$, if $b_{t_r+1} < j_{t_r}$, then denote $(c_{r+1}, d_{r+1}) = (a_{t_r+1}, b_{t_r+1})$, and if $b_{t_r+1} > j_{t_r}$, then denote $(c_{r+1}, d_{r+1}) = (a_{t_r+1} - 1, b_{t_r+1} - 1)$. We then have the following (possibly empty) valid sequence of braid moves in $\Phi(v)$:

$$v_{t_0} \xrightarrow{[c_1,d_1]} v_{t_1} \xrightarrow{[c_2,d_2]} \cdots \xrightarrow{[c_m,d_m]} v_{t_m} \ .$$

Proof: By the definition of a tag, we have $v_k \in \mathcal{R}(v)$ for each $k \in \{0, 1, \ldots, N\}$. For each $k \in \{0, 1, \ldots, N-1\}$, if $j_k \in [a_{k+1}, b_{k+1}]$, then Proposition 4.1.3 tells us v_k and v_{k+1} are exactly the same expressions. As for the case $j_k \notin [a_{k+1}, b_{k+1}]$, we have $k = t_r$ for some $r \in \{0, 1, \ldots, m\}$, and we either have $a_{t_r+1} < b_{t_r+1} < j_{t_r}$ or $j_{t_r} < a_{t_r+1} < b_{t_r+1}$. In either case, we have $j_{t_r} - 1$ is a right boundary coordinate and $j_{t_r} + 1$ is a left boundary coordinate, so it follows from Corollary 3.3.3 that $v_{t_r} \xrightarrow{[c_{r+1},d_{r+1}]} v_{t_r+1}$ is a valid sequence. Finally, we observe that v_{t_r+1} and $v_{t_{r+1}}$ are the same expression for each $r \in \{0, 1, \ldots, m-1\}$, therefore the result follows. $\square$

From the results proven above, we have justified all the assertions in our informal discussion. In particular, for any $v, w \in W$ such that $v \lhd w$, Proposition 4.1.2 tells us that the tagged letter of any $w_i \in \mathcal{R}(w)$ is not necessarily invariant. Furthermore, for $w_i, w'_i \in \mathcal{R}(w)$, even if $\P_{v \lhd w}(w_i) = \P_{v \lhd w}(w'_i) = j$ for some $j \in [\ell(w)]$, it is not necessarily true that $\tau(w_i, j) = \tau(w'_i, j)$, as the following example shows:

Example 4.1.5. Let $S = \{s, a, b, c\}$ such that $m(s, a) = 3$ and all other pairs of distinct generators commute. Let $w = sasbc$ and let $v = sabc$. Note that $v \lhd w$. We then have $sa\underline{s}bc, bc\underline{a}sa \in \mathcal{R}(w)$, where the underlined letters in each reduced expression is the tagged letter.

However, there is a special case where we do get an invariance of the tagged letter of a given fixed tag over all possible reduced expressions.

Proposition 4.1.6. Let $v, w \in W$ such that $v \lhd w$. If $\P_{v \lhd w}(w_i) = \P_{v \lhd w}(w'_i) = j$ for some $w_i, w'_i \in \mathcal{R}(w)$ and some $j \in \{1, \ell(w)\}$, then $\tau(w_i, j) = \tau(w'_i, j)$.

Proof: Denote $\ell(w) = N$, and write w_i, w'_i as $s_1 \cdots s_N$ and $s'_1 \cdots s'_N$ respectively. First consider the case $j = \ell(w)$. By the definition of a tag, we get $s_1 \cdots s_{N-1}$ and

$s'_1 \cdots s'_{N-1}$ represent the same word v, hence by multiplying v^{-1} on the left to both $s_1 \cdots s_N$ and $s'_1 \cdots s'_N$, we get s_N and s'_N represent the same word, and hence must be the same generator. The case $j = 1$ also follows from the same argument. □

Lemma 4.1.7. $D_R(vs) \subseteq D_R(v) \cup \{s\}$ for all $v \in W$ and all $s \in S \setminus D_R(v)$.

Proof: Denote w as the word vs. Since $s \notin D_R(v)$, we get $v \lhd vs = w$. Denote $\ell(v) = k$, and note that $\ell(w) = k+1$. For any $s_1 \cdots s_k \in \mathcal{R}(v)$, we have $s_1 \cdots s_k s \in \mathcal{R}(w)$, with $\P_{v \lhd vs}(s_1 \cdots s_k s)$ obviously being s by definition. Consequently, Proposition 4.1.6 tells us $\tau(w'_i, k+1) = s$ for all $w'_i \in \mathcal{R}(w)$. Now, choose an arbitrary $w_i \in \mathcal{R}(vs)$, and let $j = \P_{v \lhd vs}(w_i)$. If $j < k+1$, then $\overline{\tau}(w_i, j) \in \mathcal{R}(v)$ implies $\tau(w_i, k+1) \in D_R(v)$. If $j = k+1$, then $\tau(w_i, k+1) = s$. It then follows from Lemma 1.3.7 that $D_R(vs) \subseteq D_R(v) \cup \{s\}$. □

Theorem 4.1.8. Let $v \in W$, let $A \cap D_R(v) = \emptyset$, and let $w \in W_A$. Then we have $D_R(vw) \subseteq D_R(v) \cup A$.

Proof: We shall prove by induction on $\ell(w)$. The case $\ell(w) = 0$ is trivial, and the case $\ell(w) = 1$ is just a consequence of Lemma 4.1.7. Suppose that for some positive integer $n \geq 2$, the assertion is true for all words w having lengths $\ell(w) < n$. Now consider the case $\ell(w) = n$. Let $s_1 \cdots s_n \in \mathcal{R}(w)$, and denote u_i as the expression $s_1 \cdots s_{n-1}$, so that $u_i s_n \in \mathcal{R}(w)$. By induction hypothesis, we have $D_R(vu_i) \subseteq D_R(v) \cup A$. By Lemma 4.1.7, we get $D_R(vw) = D_R(vu_i s_n) \subseteq D_R(vu_i) \cup \{s_n\} \subseteq D_R(v) \cup A \cup \{s_n\} = D_R(v) \cup A$. Therefore, by induction, the assertion follows. □

In view of the fact that $D_R(w) = D_L(w^{-1})$ for all $w \in W$, there are analogous results to Lemma 4.1.7 and Theorem 4.1.8 for left descent sets, which we shall record down for the sake of completeness:

Corollary 4.1.9. $D_L(sv) \subseteq D_L(v) \cup \{s\}$ for all $v \in W$ and all $s \in S \setminus D_L(v)$.

Proof: This is the dual of Lemma 4.1.7. □

Corollary 4.1.10. Let $v \in W$, let $A \cap D_L(v) = \emptyset$, and let $w \in W_A$. Then we have $D_L(wv) \subseteq D_L(v) \cup A$.

Proof: This is the dual of Theorem 4.1.8. □

4.2 Dominating Descent Sets

In this section, we introduce the notion of what it means for a set to dominate another set, and relate to descent sets.

Definition. Let (W, S) be a Coxeter system. For any set $A \subseteq S$, we define

$$D(A) = \mathcal{D}_A^A = \{w \in W : D_R(w) = A\}$$

to be the set of all words in W having right descent set A.

Definition. Let (W, S) be a Coxeter system, and let $A, B \subseteq S$. If there exists an injection $\varphi : D(B) \hookrightarrow D(A)$ such that $w \leq_R \varphi(w)$ for all $w \in D(B)$, then we say A *dominates* B.

Definition. Let $A, B \subseteq S$. If $m(a,b) = 2$ for all $a \in A, b \in B$, then we say the sets A and B *commute*.

Theorem 4.2.1. Let (W,S) be a Coxeter system, and let $A, B \subseteq S$ such that A and B commute, $A \cap B = \emptyset$, and B is finite. Then $A \cup B$ dominates A, given by the injection $w \mapsto ww_0(B)$.

Proof: Choose an arbitrary $w \in D(A)$. By definition, we have $D_R(w) = A$, hence the condition $A \cap B = \emptyset$ is equivalent to $D_R(w) \cap B = \emptyset$, so Theorem 4.1.8 implies $D_R(ww_0(B)) \subseteq A \cup B$. By assumption, we have $w \in D(A)$ implies $D_R(w) \cap B = \emptyset$. Since B is finite, the largest element $w_0(B)$ in W_B exists. Proposition 2.3.6 then implies $w \leq_R ww_0(B)$. Note also that Proposition 2.4.6 implies $D_R(w_0(B)) = B$, hence $B \subseteq D_R(ww_0(B))$. Finally, since A and B commute, it follows that for every letter $s \in D_R(w) = A$, by commuting with every letter in any expression for $w_0(B)$, we can always get sequence of braid moves with the resultant word ending with the letter s, thus $A \subseteq D_R(w)$. Consequently, we have $D_R(ww_0(B)) = A \cup B$, i.e. $ww_0(B) \in D(A \cup B)$, and the result follows. $\square$

Theorem 4.2.2. Let (W,S) be a finite Coxeter system, and let $A, B \subseteq S$. If A dominates B, then $B \subseteq A$.

Proof: Suppose A dominates B, then there exists an injection $\varphi : D(B) \to D(A)$ such that $w \leq_R \varphi(w)$ for all $w \in D(B)$. Corollary 2.4.11 implies $w_0^{S\setminus B} \in D(B)$, hence $\varphi(w_0^{S\setminus B}) \in D(A)$ and $w_0^{S\setminus B} \leq_R \varphi(w_0^{S\setminus B})$ by assumption. For the sake of brevity, denote $v = \varphi(w_0^{S\setminus B})$. By the Prefix Property (Proposition 2.2.1(iv)), we can write $v = w_0^{S\setminus B} \cdot u_i$ for some reduced expression u_i. Since $v \leq w_0$, it also follows from the Prefix Property that $w_0 = v \cdot u_i'$ for some reduced expression u_i', so that we get $w_0 = w_0^{S\setminus B} \cdot u_i u_i'$. From (2.15), we have the unique factorization $w_0 = w_0^{S\setminus B} \cdot w_0(S\setminus B)$, which implies $u_i u_i' = w_0(S \setminus B)$, hence $u_i' \in W_{S\setminus B}$. Now, let $\ell(u_i') = k$, and write u_i' as the reduced expression $s_{i_1} \cdots s_{i_k}$. For each $t \in [k]$, denote $v_t = vs_{i_1} \cdots s_{i_t}$, and denote $v_0 = v$. By definition, we have $s_{i_t} \notin D_R(v_{t-1})$ for every $t \in [k]$, hence applying Lemma 4.1.7 inductively, we get $D_R(v_k) \subseteq D_R(v_0) \cup \{s_{i_1}, \ldots s_{i_k}\}$. Since $u_i' \in W_{S\setminus B}$, we have $\{s_{i_1}, \ldots s_{i_k}\} \subseteq S \setminus B$. Note that $v_0 = v \in D(A)$ by definition, so $D_R(v_0) = A$. Finally, $D_R(w_0) = S$ by Proposition 2.4.3, hence we get $S \subseteq A \cup (S \setminus B)$, which implies $B \subseteq A$. $\square$

Chapter 5

Applications

In this chapter, we give some applications of the new results we have derived in Chapter 4. Each section gives a very brief exposition of the background needed to be able to state the corresponding applications. Given the scope of this paper, it is impossible to develop the theory in each of the expositions in full detail, so our purpose is merely to give a flavor of how our results on Coxeter systems, in particular on dominating sets, can be applied to other areas. Often, the reader will be referred to the references for more details.

In order not to lose track of the main ideas involved in these applications, we shall assume the reader is familiar with notions of posets, lattices, matroids, simplicial complexes, order complexes, Cohen-Macaulay complexes, as well as other notions related to their usage. The unfamiliar reader is urged to see [BB04] for a crash-course on the relevant definitions and notations.

5.1 Geometric Lattices and Flag h-vectors

In this section, we focus our attention on the standard Coxeter system (S_{d+1}, S) that was discussed in Chapter 1.1, with $S = \{s_1, \ldots, s_d\}$, s_i being the transposition $(i, i+1)$ for each $i \in [d]$. For brevity, each subset $A = \{s_{i_1}, \ldots, s_{i_k}\} \subseteq S$ can be simply denoted as $\{i_1, \ldots, i_k\} \subseteq [d]$. With the simplification in notation, we refer to descents sets of subsets of $[d]$, i.e. for $A \subseteq [d]$, the descent set of A,

$$D(A) = \mathcal{D}_A^A = \{w \in S_{d+1} : D_R(w) = A\}, \tag{5.1}$$

is the set of all elements in Coxeter group S_{d+1} having right descent set A. The notion of "A dominates B" for sets $A, B \subseteq [d]$ can also be defined analogously, and we shall implicitly assume that any statements made about descent sets in this section is with respect to this Coxeter system (S_{d+1}, S).

Given a finite $(d-1)$-dimensional abstract simplicial complex Δ, one of the most fundamental combinatorial invariants is its f-vector, which is a sequence of integers $(f_0, f_1, \ldots, f_d)$, where for each $i \in \{0, 1, \ldots, d\}$, f_i denotes the number of i-dimensional faces of Δ. The h-vector of Δ is defined to be the sequence $(h_0, h_1, \ldots, h_d)$, where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_j(\Delta). \tag{5.2}$$

The h-vector of Δ is derived from the corresponding f-vector by an invertible transformation, and given the h-vector, we can always derive the f-vector via the following identity:

$$f_j(\Delta) = \sum_{i=0}^{j} \binom{d-j}{d-i} h_i(\Delta). \tag{5.3}$$

In view of the above correspondence, knowing information about the f-vector is equivalent to knowing information about the h-vector. One advantage of studying h-vectors is that certain properties of f-vectors are more easily expressed in terms of the h-vector. An interesting example is the notion of a convex ear decomposition, first introduced by Chari [Cha97]:

Definition. Let Δ be a pure $(d-1)$-dimensional simplicial complex. A *convex ear decomposition* of Δ is an ordered sequence $\Delta_1, \Delta_2, \ldots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ satisfying the following:

(i) Δ_1 is the boundary complex of a simplicial d-polytope.

(ii) For each $j \in \{2, \ldots, m\}$, Δ_j is a $(d-1)$-ball which is a proper subcomplex of the boundary of a simplicial d-polytope.

(iii) $\Delta_j \cap \left(\bigcup_{k=1}^{j-1} \Delta_k \right) = \partial \Delta_j$ for $j \geq 2$.

(iv) $\Delta = \displaystyle\bigcup_{k=1}^{m} \Delta_k$.

Δ_1 is called the *initial* subcomplex, while Δ_j, for each $j \geq 2$, is called an *ear* of the decomposition.

The following theorem gives the link between convex ear decompositions and h-vectors:

Theorem 5.1.1. Let Δ be a $(d-1)$-dimensional simplicial complex. If Δ has a convex ear decomposition, then for all $i \leq \lfloor \frac{d}{2} \rfloor$, the h-vector of Δ satisfies:

$$h_{i-1} \leq h_i, \tag{5.4}$$
$$h_i \leq h_{d-i}. \tag{5.5}$$

Proof: See [Cha97]. □

In [NS04], Nyman and Swartz proved that the order complex of a geometric lattice has a convex ear decomposition, hence an immediate consequence of Theorem 5.1.1 is the following:

Theorem 5.1.2. Let L be a rank $(d+1)$ geometric lattice and let $\Delta(L)$ be the order complex of L. Then for all $i \leq \lfloor \frac{d}{2} \rfloor$, the h-vector of $\Delta(L)$ satisfies the following:

$$h_{i-1} \leq h_i, \tag{5.6}$$
$$h_i \leq h_{d-i}. \tag{5.7}$$

Proof: See [NS04]. □

We remark that the theorem in [NS04], which states the order complex of a geometric lattice has a convex ear decomposition, was later extended by Schweig in [Sch08] to include rank-selected subposets of geometric lattices, and we have the more general theorem:

Theorem 5.1.3. Let L be a rank d geometric lattice and let $S \subseteq [d-1]$. Then the order complex $\Delta(\overline{L}_S)$ admits a convex ear decomposition.

Proof: See (Theorem 3.13, [Sch08]). □

A very related concept of the h-vectors is the notion of flag h-vectors. Similar to the case of h-vectors, we first define flag f-vectors, and we then define the flag h-vectors in terms of the flag f-vectors.

Definition. Let Δ be a $(d-1)$-dimensional complex. A *flag* of faces in Δ is a chain $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of faces F_i in Δ. For any set $S \subseteq [d-1]$, a flag is an *S-flag* if

$$S = \{\dim F_1, \dim F_2, \ldots, \dim F_k\}. \tag{5.8}$$

Denote f_S as the number of S-flags in Δ. We say the function $S \mapsto f_S$ (for $S \subseteq [d-1]$) is the *flag f-vector* of Δ. Also, define

$$h_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T. \tag{5.9}$$

We then say the function $S \mapsto h_S$ (for $S \subseteq [d-1]$) is the *flag h-vector* of Δ.

The paper by Björner [Bjo80] provides the link between geometric lattices and descent sets, where he proved that for P a graded poset admitting an R-labelling, $h_S(P)$ is the number of maximal chains of P with labels having descent set S. (See Theorem 2.7, [Bjo80].) Nyman and Swartz then used Björner's result to prove the following theorem:

Theorem 5.1.4. Let L be a rank d geometric lattice and let $\Delta(L)$ be the order complex of L. Let $A, B \subseteq [d-1]$. If A dominates B, then the flag h-vector of $\Delta(\overline{L})$ satisfies $h_B \leq h_A$.

Proof: See [NS04]. □

The relation of Nyman and Swartz's work with our work will be discussed in Section 5.4.

5.2 Supersolvable Lattices with Nowhere-zero Möbius Function

In his PhD thesis [Sch08], Schweig studied the rank-selected subposets of supersolvable lattices with nowhere-zero Möbius function. Two notable special cases of such posets are the super-solvable lattices with nowhere-zero Möbius function, and the rank-selected subposets of Boolean lattices.

Definition. Let $P = \{x_1, \dots, x_d\}$ be a finite poset with $|P| = d$ for some $d \in \mathbb{Z}^+$. An *order completion* of P is a total ordering of its elements, so that $x_i < x_j$ implies $i < j$ for all $i, j \in [d]$. An *order ideal* of P is a subset $I \subseteq P$ such that $y \in I$ and $x < y$ implies $x \in I$. Denote $\mathcal{I}(P)$ as the poset of order ideals of P ordered by inclusion. A finite lattice L is said to be *distributive* if there exists a poset P such that L is isomorphic to $\mathcal{I}(P)$. For any lattice L, L is said to be *supersolvable* if there exists a maximal chain c_M of L, called the *M-chain*, such that the sublattice of L generated by c_M and any other (not necessarily maximal) chain of L is a distributive lattice.

Definition. A poset P is said to have a *nowhere-zero Möbius function* μ if $\mu(x, y) \neq 0$ whenever $x, y \in P$ and $x \leq y$

Definition. The rank d *Boolean lattice*, denoted by B_d, is the poset of all subsets of $[d]$ ordered by inclusion.

Motivated by the application of Chari's result [Cha97] to geometric lattices as done in [NS04], Schweig proved the following theorem, hence getting the next two corollaries as special cases:

Theorem 5.2.1. Let L be a rank d supersolvable lattice with nowhere-zero Möbius function, and let $S \subseteq [d-1]$. Then the order complex $\Delta(\overline{L}_S)$ admits a convex ear decomposition.

Proof: See Section 2 in [Sch08] (the proof of this result is split over a few theorems). □

Corollary 5.2.2. Let L be a rank d supersolvable lattice with a nowhere-zero Möbius function. Then the order complex $\Delta(\overline{L})$ admits a convex ear decomposition.

Proof: This follows immediately from Theorem 5.2.1. □

Corollary 5.2.3. Let $S \subseteq [d-1]$. The order complex $\Delta((\overline{B}_d)_S)$ admits a convex ear decomposition.

Proof: A Boolean lattice B_d is an example of a supersolvable lattice with a nowhere-zero Möbius function, so this follows from Theorem 5.2.1. (In fact, it is a distributive lattice.) See [Sch08] for a discussion on boolean lattices. □

Just as in the case of geometric lattices (Theorem 5.1.2 above), Chari's result [Cha97] implies the h-vector inequalities for each of the three classes of posets mentioned above. Using similar techniques as in [NS04], Schweig concluded the following:

Theorem 5.2.4. Let L be a rank d supersolvable lattice with a nowhere-zero Möbius function, and let $A, B \subseteq [d-1]$. If A dominates B, then the flag h-vector of the order complex $\Delta(\overline{L})$ satisfies $h_B \leq h_A$.

Proof: See (Theorem 4.2.2, [Sch08]). □

5.3 Face Posets of Cohen-Macaulay Simplicial Complexes

As a continuation of the previous section, Schweig also studied the face posets of Cohen-Macaulay simplicial complexes in his PhD thesis [Sch08], where similar to the other results he obtained, he also proved that the order complexes of such face posets admit convex ear decomposition, thereby also getting the flag h-vector inequalities for this class of posets by applying Chari's result [Cha97]. Although his proof is very similar to the case of supersolvable lattices with nowhere-zero Möbius function, the key difference is that he used the additional ingredient of Hibi's result [Hib88], which states that the codimension-1 skeleton of a shellable complex is 2-Cohen-Macaulay.

Theorem 5.3.1. Let Σ be a d-dimensional shellable complex with face poset P_Σ, and let $S \subseteq [d-1]$. Then the order complex $\Delta((\overline{P}_\Sigma)_S)$ admits a convex ear decomposition.

Proof: See (Theorem 3.2.1, [Sch08]). [Note that there is a printing error in the statement of Theorem 3.2.1 in [Sch08]. Σ should be a d-dimensional complex, not a $(d-1)$-dimensional complex as stated in [Sch08].] □

Theorem 5.3.2. Let Σ be a d-dimensional shellable complex with face poset P_Σ, and let $A, B \subseteq [d-1]$. If A dominates B, then the flag h-vector of the order complex $\Delta(\overline{P}_\Sigma)$ satisfies $h_B \leq h_A$.

Proof: See (Theorem 4.2.4, [Sch08]). □

Schweig also generalized the above result as follows:

Theorem 5.3.3. Let K be a d-dimensional Cohen-Macaulay simplicial complex with face poset P_K, and let $A, B \subseteq [d-1]$. If A dominates B, then the flag h-vector of the order complex $\Delta(\overline{P}_K)$ satisfies $h_B \leq h_A$.

Proof: See (Theorem 4.2.5, [Sch08]). □

As remarked in [Sch08], Theorem 5.3.3 cannot be extended to include posets whose order complexes are Cohen-Macaulay, and Schweig considered the order complex of a Gorenstein* poset as a counter-example.

5.4 Relation to our Work

As can be seen from the previous three sections, the order complexes corresponding to geometric lattices, supersolvable lattices with nowhere-zero Möbius function, and face posets of Cohen-Macaulay simplicial complexes, all have flag h-vectors satisfy the condition that A dominates B implies $h_B \leq h_A$. In all these results, the main argument is the same: Show that the corresponding order complex admits a convex ear decomposition, apply Chari's result [Cha97] (Theorem 5.1.1 above), then use the technique of minimal labelling as discussed in [NS04] to obtain the desired conclusion.

We remark that Chari's result (Theorem 5.1.1 above) was proven using a deep result by Stanley [Sta80], which involves the hard Lefschetz Theorem from algebraic geometry. This means all the results in the previous three sections are indirectly dependent on the Lefschetz Theorem. It would be very desirable to be able to give a combinatorial proof to the inequalities of the h-vector and avoid using the Lefschetz Theorem.

It is then with the motivation of Nyman and Swartz's result on the relation between dominating descent sets of order complexes and the flag h-vector inequalities that we study the descent sets of general Coxeter systems, hoping to get a complete characterization of when A dominates B via a combinatorial proof. If we can get such a characterization in the general case of Coxeter systems, then applying to the Coxeter groups of type A_n, there is an implied combinatorial proof at least for the flag h-vector inequalities, without having to rely on the Lefschetz Theorem.

Although we did not give a complete characterization in this paper, we did get a partial characterization in Theorem 4.2.1, where we give an explicit map for $A \cup B$ to dominate B, in the case when A, B are disjoint commuting sets, with B finite. We also remark that we proved in Theorem 4.2.2 that for all finite Coxeter systems, if A dominates B, then $B \subseteq A$. This is a generalization of Proposition 5.4 in [NS04], which is the special case of our result for Coxeter systems of type A_n.

5.5 Finite Buildings

For all the results discussed in sections 5.1, 5.2 and 5.3, we have analogous results of A dominates B implies $h_B \leq h_A$ for the various classes of lattices and posets. We observe that in each case, the descent sets involved in A dominating B involve Coxeter systems of type A_n. In this section, we shall look at finite buildings.

Definition. Let (W, S) be a Coxeter system. Let $\Sigma(W, S)$ be the poset of standard cosets in W, ordered by reverse inclusion. If $A \subseteq B$ as subsets of W, then we say B is a *face* of A. We define $\Sigma(W, S)$ to be the *Coxeter complex* associated to the Coxeter system (W, S). The elements of $\Sigma(W, S)$ are called *simplices*, and the maximal simplices (i.e. the singletons $\{w\}$) are called *chambers* and are identified with their corresponding elements in W they contain.

Definition. The Coxeter complex $\Sigma(W, S)$ is called *spherical* if it is finite, or equivalentlt, if W is finite.

Definition. Let $\Sigma(W, S)$ be a finite Coxeter complex. A *finite building* of type (W, S) is a (finite) simplicial complex Δ that is the union of subcomplexes Σ, called *apartments*, such that the following hold:

(i) Each apartment Σ is isomorphic to $\Sigma(W, S)$.

(ii) For any two faces ρ_1 and ρ_2 in Δ, there is an apartment Σ containing both of them.

(iii) If Σ and Σ' are two apartments containing ρ_1 and ρ_2, then there is an isomorphism $\Sigma \to \Sigma'$ fixing ρ_1 and ρ_2 pointwise.

In 2006, Swartz considered finite buildings in [Swa06], and extended the ideas involved in [NS04] to prove the following result:

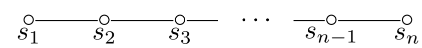
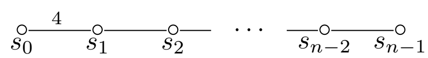
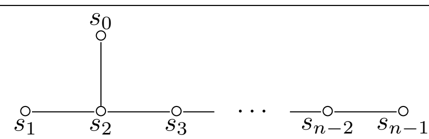
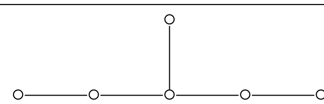
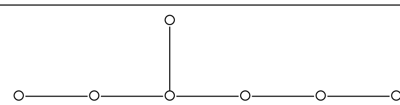
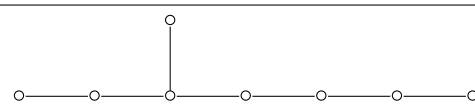
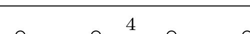
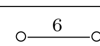
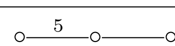
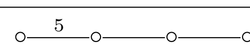
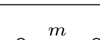
Theorem 5.5.1. Let (W, S) be a finite Coxeter system. Let A and B be subsets of S, and assume A dominates B. If Δ is a finite building of type (W, S), then $h_B \leq h_A$.

Proof: See (Theorem 2.4, [Swa06]) □

This is the first case where the descent sets involved in A dominating B involve finite Coxeter systems, and not just the specific case of Coxeter systems of type A_n. It is known that for Coxeter systems of type A_n, we have A dominates B implies $B \subseteq A$, however it was previously not known whether this result could be extended to general Coxeter systems. In this paper, we have proven in Theorem 4.2.2 that the result can indeed be extended to finite Coxeter systems, hence in the assumptions for Theorem 5.5.1, A dominates B necessarily implies $B \subseteq A$ by our result.

Appendix A

Classification of Finite Irreducible Coxeter Systems

Name	Coxeter Diagram
A_n (for $n \geq 1$)	
B_n (for $n \geq 2$)	
D_n (for $n \geq 4$)	
E_6	
E_7	
E_8	
F_4	
G_2	
H_3	
H_4	
$I_2(m)$ (for $m \geq 3$)	

The corresponding Coxeter groups are pairwise non-isomorphic, with exceptions:

$$I_2(3) = A_2, I_2(4) = B_2, I_2(6) = G_2.$$

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